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DENSITY IN HYPERBOLIC SPACES

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DENSITY IN HYPERBOLIC SPACES

by

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Dedicated to Jaymee and Ryan

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DENSITY IN HYPERBOLIC SPACES

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We present new (and old) examples showing the difficulty of defining density for packings of hyperbolic space. Using probabilistic techniques, we develop a new method for studying density and show that it corresponds to well-founded notions in Euclidean space. Using this new machinery, we prove a conjecture of G. Fejes Toth, G. Kuperberg and W. Kuperberg regarding the existence of locally densest packings. We also prove that for most radii r all optimally dense packings of hyperbolic space by spheres of radius r have low symmetry.

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Chapter 1

Introduction

For over fifty years, researchers (L. Fejes-Toth [30], Boroczky-Florian [7], Boroczky [6]) have attempted to extend the theory of densest sphere packings in Euclidean space to hyperbolic space. They encountered various difficulties, one of which we describe below. Because of these difficulties, it was generally concluded that there is no convincing way to define density for sphere packings of hyperbolic space [4], [6], [31], [26], [12]. To avoid these problems, several alternatives to the notion of densest packings were introduced [32], [12], [16], one of which will be explained below.

Using techniques from ergodic theory and probability theory [2], [20], Charles Radin and I developed a framework through which density can be defined for a large class of hyperbolic sphere packings [9]. This new definition is compatible with usual methods for defining density in Euclidean space. These new results provide a foundation through which dense sphere packings in hyperbolic space may be studied. Using similar techniques, I proved a conjecture of G. Fejes Toth, G. Kuperberg and W. Kuperberg [12] regarding the existence of locally densest packings, thereby proving the strength of this new framework.

1.0.1 An example

Consider the following example: in the upperhalf space model of the hyperbolic plane let $\mathcal{P}$ be the circle packing shown in figure 1.2.

All the circles of $\mathcal{P}$ are hyperbolically congruent and the set of Euclidean centers of $\mathcal{P}$ is equal to $\{2^{2k+1}x + 2^{2k} + 2^{2k}3i \,|\, x, k \in \mathbb{Z}\}$. The Euclidean rectangle R_l in light grey on the left is equal to the union of three Euclidean rectangles all of which are hyperbolically congruent. Its images under the maps $z \to 4^k(z + 2x)$ (for $k, x \in \mathbb{Z}$), forms a tiling in which each tile contains exactly one circle of the packing. The Euclidean rectangle R_r in dark grey on the right is isometric to R_l but it contains two circles of the packing. Its images under the maps $z \to 4^k(z + x)$ (for $k, x \in \mathbb{Z}$) forms a tiling in which each tile contains exactly two circles of the packing. Hence the density of the packing with respect to the first tiling must be half the density of the packing with respect to the second tiling. Because of examples such as this, it was generally concluded [Bo1, FeK] that there is no reasonable way to define the density of packings in hyperbolic spaces.

In order to illustrate exactly what can go wrong, in section 1.2 we present examples of packings of hyperbolic space that do not have well-defined densities.

1.0.2 Invariant Measures

Let $\Sigma_{\mathcal{B}}$ be the set of packings (or tilings) of hyperbolic space $\mathbb{H}^n$ by a collection $\mathcal{B}$ of bodies (which could be sphere or polyhedra, for example), with $\Sigma_{\mathcal{B}}$ topologized so that two packings are close if they are close in the Hausdorff sense in a large ball centered at the origin. $\Sigma_{\mathcal{B}}$ is a compact space on which the isometry group of $\mathbb{H}^n$ acts continuously. The space $\mathcal{M}$ of Borel probability measures on $\Sigma_{\mathcal{B}}$ that are invariant under this action is also a compact space under the weak* topology. All the invariant measures are convex sums of ergodic measures. The ergodic measures define a class of packings whose members are all locally congruent [9]. Thus the ergodic measures serve as a natural substitute for individual packings. In chapter 2 we explain the use of invariant measures. In section 2.1 we put the study of dense packings on a firm foundation by proving that these measures are compatible with all the usual notions of density in Euclidean space.

The main result of chapter 3 is that the set of radii r for which there exists optimally dense packings by spheres of radius r with high symmetry is at most countable. Thus for most r, all densest packings by balls of radius r (in hyperbolic space) have low symmetry. We do not yet know how asymmetric a densest sphere packing can be. Indeed, there are no explicit examples known of asymmetric densest sphere packings.

In [12] G. Fejes Toth, G. Kuperberg and W. Kuperberg introduced the notion of a completely saturated packing, that is a packing for which it is impossible to replace a finite number of bodies of the packing with a finite number of bodies of greater total volume without introducing overlaps. Intuitively, this notion means locally densest. They conjectured that for any body in either Euclidean or hyperbolic space, a completely saturated packing by that body exists. In section 2.2.1 we prove this conjecture in full generality. In fact the proof extends to multiple bodies, to more general spaces and to more general phenomena than packing. Also we show in section 2.2.2 that completely saturated packings in hyperbolic space need not be densest packings contradicting what was predicted in [12].

A major difference between the Euclidean and the hyperbolic case is that in the latter case invariant measures do not exist on every invariant closed subset of $\Sigma_{\mathcal{B}}$. The difficulties earlier researchers encountered can be explained by the absence of such measures. In section 2.3 we present various examples of this phenomenon.

We have distributed results of this thesis in preprints [9], [10], [8].

1.1 Background on Hyperbolic Geometry

In this section, we set the notation for the rest of the paper while providing background on hyperbolic geometry. For more background, see [24] or [1].

Hyperbolic space $\mathbb{H}^n$ is the unique simply connected Riemannian manifold of constant sectional curvature -1. There are several useful models of $\mathbb{H}^n$ but we will only use the upperhalf space model. Therefore, we identify $\mathbb{H}^n$ with $\{(x_1, .., x_n) \in \mathbb{E}^n | \, x_n > 0\}$ under the Riemannian metric given by $ds^2 = dx^2/y^2$. In other words, if v and w are vectors at a point p then the hyperbolic inner product is given by $< v, w >_{\mathbb{H}} = \frac{<v,w>}{p_n^2}$ where $< \cdot, \cdot >$ denotes the Euclidean inner product and p_n is the n^{th} coordinate of p. The distance function d on $\mathbb{H}^n$ can be explicitly computed as

$$\cosh(d[(x,y),(v,w)]) = 1 + \frac{|x-v|^2 + |y-w|^2}{2yw} \tag{1.1}$$

where x and v are elements of $\mathbb{E}^{n-1}$ and y and w are positive real numbers. The volume of a subset $E \subset \mathbb{H}^n$ can be computed by

$$vol(E) = \int_E 1/p_n^n \, dp. \tag{1.2}$$

In two dimensions, the Gauss-Bonnet theorem gives an easy formula for computing the area of polygons. If the vertices of the polygon P are given by $v_1, ..., v_n$ and their interior angles are $a(v_1), .., a(v_n)$, then

$$area(P) = (n-2)\pi - \Sigma_{i=1}^{n} a(v_i). \tag{1.3}$$

We will have occasion to use the following three equations from hyperbolic trigonometry. The first two equations are called the "law of cosines" and the second is the "law of sines". Suppose we are given a triangle with vertices T, U and V. Then denoting the length of the segment TU by TU and the angle at T by $\angle UTV$ (for example), we obtain

$$\cosh(TU) = \frac{\cos(\angle UTV)\cos(\angle VUT) + \cos(\angle TVU)}{\sin(\angle UTV)\sin(\angle VUT)} \tag{1.4}$$

$$\cos(\angle TUV) = \frac{\cosh(TU)\cosh(UV) - \cosh(VT)}{\sinh(TU)\sinh(UV)} \tag{1.5}$$

$$\frac{\sin(\angle UTV)}{\sinh(VU)} = \frac{\sin(\angle TVU)}{\sinh(UT)} = \frac{\sin(\angle VUT)}{\sinh(TV)}. \tag{1.6}$$

Perhaps one of the most important facts we will need about hyperbolic space is that the growth rate of the volume of a sphere is exponential with respect to its radius.

Theorem 1.1.1. *(see [11]) There exists positive real constants c_1 and c_2 (depending on the dimension n) such that*

$$\lim_{R\to\infty} vol(B_R(p))e^{-c_1 R} = c_2. \tag{1.7}$$

where $B_R(p)$ denotes the ball of radius R centered at a point $p \in \mathbb{H}^n$.

In the 2-dimensional case, we have $vol(B_R(p)) = 2\pi(\cosh(R) - 1)$.

We discuss next the group $\mathcal{G}$ of orientation-preserving isometries of $\mathbb{H}^n$. If we let $\mathcal{O}$ be a point on $\mathbb{H}^n$ then the stabilizer of $\mathcal{O}$ in $\mathcal{G}$ is isomorphic to $SO(n-1)$ for it acts transitively and faithfully on the set of ordered bases of the tangent space of $\mathcal{O}$ (with a given orientation). If we let $\pi : \mathcal{G} \to \mathbb{H}^n$ be the map defined by $\pi(g) = g\mathcal{O}$ then π is onto, so we may identify $\mathbb{H}^n$ with $\mathcal{G}/SO(n-1)$. In the upper-half space model any Euclidean similarity that preserves the plane $\{(x_1, .., x_{n-1}, 0) \in \mathbb{E}^n\}$ is a hyperbolic isometry.

There is a unique (up to scalars, see [17]) measure $\lambda_{\mathcal{G}}$, called Haar measure, on $\mathcal{G}$ that is invariant under both left and right multiplication by $\mathcal{G}$. In other words, for every Borel subset $E \subset \mathcal{G}$ and $g \in \mathcal{G}$ $\lambda_{\mathcal{G}}(E) = \lambda_{\mathcal{G}}(gE) = \lambda_{\mathcal{G}}(Eg)$. We also have $\lambda_{\mathcal{G}}(E) = \lambda_{\mathcal{G}}(E^{-1})$. This property is called unimodularity. We will assume that $\lambda_{\mathcal{G}}$ has been scaled so that for every (Borel) subset $E \subset \mathbb{H}^n$, $vol(E) = \lambda_{\mathcal{G}}(\pi^{-1}(E))$. For a proof that this is possible, see [24].

In the upperhalf space model, the set $\Delta = \{p \in \mathbb{E}^n | p_n = 0\} \cup \infty$ with the topology of the $n-1$-sphere is called the sphere at infinity. The action of $\mathcal{G}$ on $\mathbb{H}^n$ extends continuously (in fact conformally) to this sphere.

A horosphere is a codimension 1 subset of $\mathbb{H}^n$ that is conformally equivalent to $\mathbb{E}^{n-1}$. In the upper-half space model, horospheres come in two types, those that are "centered at ∞" which are Euclidean planes parallel to Δ and those that are centered at p for $p \in \Delta - \infty$ which are equal to Euclidean spheres tangent to Δ at p. A horosphere is, in a sense, a limit of spheres. For example, if S_R is the sphere centered at $(0, .., 0, e^R)$ of hyperbolic radius R then S_R limits onto the plane $x_n = 1$ as $R \to \infty$ in the sense of uniform convergence on compact sets.

1.2 How density in Hyperbolic Space is different than density in Euclidean space

We will be analyzing the density of certain subsets of Euclidean n-dimensional space $\mathbb{E}^n$ or hyperbolic n-dimensional space $\mathbb{H}^n$ of curvature -1; we let $\mathbb{S}$ stand for any of these spaces. The subsets of $\mathbb{S}$ whose density we will consider will generally be "packings" of (infinitely many) "bodies" β_j, where a packing is a collection of bodies with pairwise disjoint interiors, and a body is a compact, connected set which is the closure of its interior. One of the features of our analysis will be an emphasis on distinguishing between the density of the packing versus the density of the set which is the union of the bodies in the packing; that is, it will be significant to maintain the individuality of each of those bodies.

Our main focus will be on the "densest" packings possible by the given bodies, and this requires examination of the primitive notion of density. If we were packing a region S of *finite* volume by the bodies β_j, the density of such a packing would be unambiguous – the fraction of the volume of S covered by the bodies – but density must be defined more subtly for packings of a region, such as $\mathbb{S}$, of infinite volume. The most widely accepted [26] primitive notion is that the density of a packing $\mathcal{P}$ of $\mathbb{S}$ should be obtainable by choosing a family of finite volume regions S_k, with $S_k \subset S_{k+1}$ and $\cup_k S_k = \mathbb{S}$, and the density of $\mathcal{P}$ should be

$$\lim_{k\to\infty} \frac{vol(\mathcal{P} \cap S_k)}{vol(S_k)}, \tag{1.8}$$

where $vol(\cdot)$ denotes volume in $\mathbb{S}$ and $\mathcal{P} \cap S_k$ denotes the portion of S_k covered by bodies in $\mathcal{P}$. We would want the density to be reasonably independent of the family S_k.

It is worth noting that the limit in (1.8) can easily fail to exist. Consider the sequence $\{D_j \,|\, j \geq 1\}$ of closed disks in $\mathbb{E}^2$, D_j of radius 2^j and centered at the origin. Let P_j be the annulus D_j/D_{j-1} between successive disks, and let S be the union of those P_j with $j \geq 2$ even. If we try to define the density of S using the expanding regions $S_k = D_k$, the sequence of local or approximate densities $vol(S \cap S_k)/vol(S_k)$ would not have a limit as $k \to \infty$, due to oscillation. (We could get the same qualitative result by replacing our region S by its intersection with some simple packing of disks, such as the packing of unit diameter disks whose centers have integer coordinates.)

Even though there are packings without a well defined density there is no real difficulty in defining *optimal* density of packings in Euclidean space. In fact we now show how to construct densest packings of Euclidean space. Let $\mathbb{S} = \mathbb{E}^n$ and let S_k be a cube centered at the origin, with edges of length k aligned with the axes. For any $k > 0$, let $\mathcal{P}_k$ be a packing by $\mathcal{B} \equiv \{\beta_j\}$ such that all bodies in $\mathcal{P}_k$ intersect S_k and $vol(\mathcal{P}_k \cap S_k)$ is optimally large. (Such a packing is easily shown to exist by a simple compactness argument [15].) For any packing $\mathcal{P}$ in $\mathbb{S}$ we define

$$d_k(\mathcal{P}) = \frac{vol(S_k \cap \mathcal{P})}{vol(S_k)} \tag{1.9}$$

$$d_k = \max_{\mathcal{P}} d_k(\mathcal{P}) \tag{1.10}$$

$$d = \limsup_{k\to\infty} d_k. \tag{1.11}$$

($\lim_{k\to\infty} d_k$ exists but we do not need this fact.)

At this point it is convenient to have a space $\Sigma_{\mathcal{B}}$ of all possible packings of $\mathbb{S}$ by the bodies β_j, equipped with a metric topology such that a sequence of packings converges if it converges uniformly on compact subsets of $\mathbb{S}$. We will spell this out in 2, but assume for now such a space makes sense and is in fact compact. Then we let $\mathcal{P}_\infty$ be an accumulation point of $\{\mathcal{P}_k\}$.

The following is a simple observation.

Lemma 1.2.1. *$d_k(g\mathcal{P}_\infty) \to d$ as $k \to \infty$ for every fixed rigid motion g.*

Proof. The main estimates needed are the simple facts, for $k' > k$:

$$vol(\mathcal{P}_k \cap S_k) \geq vol(\mathcal{P}_{k'} \cap S_k) \tag{1.12}$$

$$vol(\mathcal{P}_{k'} \cap S_k) \geq vol(\mathcal{P}_k \cap S_k) - [k^n - (k-C)^n], \tag{1.13}$$

where C is larger than the diameter of any body in $\mathcal{B}$. The latter holds because if it did not one could arrive at a contradiction by altering $\mathcal{P}_{k'}$ as follows. First replace the bodies of $\mathcal{P}_{k'}$ that are completely contained in S_k by the bodies of $\tilde{\mathcal{P}}_k$ that do not overlap the other bodies of $\tilde{P}_{k'}$ (i.e. that do not overlap any body of $\tilde{P}_{k'}$ that overlaps the complement of S_k). Note that the volume of bodies of $\tilde{P}_k$ that we have introduced is at least as large as the right hand side of equation 1.13. Since $vol(\mathcal{P}_{k'})$ is as large as possible, this operation could not have increased its volume. This proves equation 1.13. Since $\mathcal{P}_\infty$ is a limit of $\mathcal{P}_k$ we get that (1.13) holds if $\mathcal{P}_{k'}$ is replaced by $\mathcal{P}_\infty$.

Finally, if k_m is a sequence such that $d_{k_m} \to d$ as $m \to \infty$:

$$\begin{aligned} |d - d_{k_m}(g\mathcal{P}_\infty)| &\leq |d_{k_m} - d| + |d_{k_m} - d_{k_m}(g\mathcal{P}_\infty)| && (1.14) \\ &= |d_{k_m} - d| + |d_{k_m}(\mathcal{P}_{k_m}) - d_{k_m}(g\mathcal{P}_\infty)| && (1.15) \end{aligned}$$

and $|d_{k_m}(\mathcal{P}_{k_m}) - d_{k_m}(g\mathcal{P}_\infty)| \to 0$ as $m \to \infty$ from (1.13). $\square$

Thus, in Euclidean space optimally dense packings $\mathcal{P}$ exist for any collection $\mathcal{B}$ in the sense that their density defined by (1.8) exists, is independent of the center of expansion of (1.8), and is as large as that for any packing.

As we shall see, the above technique does not extend to $\mathbb{S} = H^n$ and therefore some other method must be used to define optimal density in $\mathbb{H}^n$. Before exhibiting such a method, we present some examples to highlight some differences between hyperbolic and Euclidean packings.

1.2.1 half-space

Consider the half space region S, defined, in the upper half plane model of the hyperbolic plane, as the set of points (x, y) with $x \geq 0$. If we try to define the density of this region by circles all expanding about a common center c, it is easy to see that the density would depend on c, with any value strictly between 0 and 1 being obtainable for appropriate c. This suggests that we will want the origin, used for the expanding regions in (1.8), to be arbitrary.

1.2.2 stripe model

We now give a simple example of a region S in the hyperbolic plane such that, when we try to define the density of S relative to a sequence of circles expanding about some point, we get the kind of oscillation we found in the Euclidean annulus example. We define the "stripe model" in the (upper half plane model of the) hyperbolic plane, where the stripes are the regions separated by the horocycles h_j, $j \in Z$, defined by $y = y_j \equiv e^{(j+1/2)W}$, where fixed $W >> 1$ is to be specified. These curves are equidistant by W in the hyperbolic metric. We call those stripes separated by h_{2j} and h_{2j+1} "black", and the others white, and we declare the region S of interest to be the union of the black stripes.

Consider the circle with hyperbolic center $c = (0, 1)$ and hyperbolic radius $R = (N + 1/2)W$, where $N >> 1$ is to be specified. We will use the following relations between the hyperbolic center (H, K) and hyperbolic radius R of a given circle and its Euclidean center (h, k) and Euclidean radius r:

$$h = H, \quad k^2 - r^2 = K^2, \quad r = k \tanh(R). \tag{1.16}$$

So our circle has Euclidean center $(0, \cosh[R])$ and Euclidean radius $\sinh(R)$.

We will show that, if N is even, the area inside the circle, of the black stripes is larger than that of the white stripes; in particular, each black stripe, between h_j and h_{j+1}, $j \leq N - 3$, is larger (by a factor 2) than that of the neighboring white stripe above it (between h_{j+1} and h_{j+2}), and therefore the area of the circle is at least 2/3 black.

For $-N - 1 \leq j \leq N - 1$, the area A_j of the stripe between h_j and h_{j+1} is:

$$A_j = \int_{y_j}^{y_{j+1}} \int_{-[2y\cosh(R)-1-y^2]^{\frac{1}{2}}}^{[2y\cosh(R)-1-y^2]^{\frac{1}{2}}} \frac{1}{y^2}\, dx\, dy \tag{1.17}$$

$$= \int_{y_j}^{y_{j+1}} \frac{2[2y\cosh(R)-1-y^2]^{\frac{1}{2}}}{y^2}\, dy. \tag{1.18}$$

For $-N \leq j \leq N-2$ the leading behavior as $N,\ W \to \infty$ (and recalling that $R = [N+1/2]W$), is

$$A_j \sim \int_{y_j}^{y_{j+1}} \frac{2y^{1/2}e^{R/2}}{y^2}\, dy \tag{1.19}$$

$$\sim 4e^{[R/2-(j+1/2)W/2]} \tag{1.20}$$

where $a \sim b$ means $\dfrac{a}{b} \to 1$ as $N,\ W \to \infty$. So

$$\frac{A_j}{A_{j+1}} \sim e^{W/2}. \tag{1.21}$$

For $j = -N-1$ we have:

$$A_{-N-1} = \sim \int_{e^{-R}}^{e^{-R+W}} \frac{2(e^R y - 1)^{1/2}}{y^2}\, dy \tag{1.22}$$

$$\sim 2e^R \int_1^{e^W} \frac{(z-1)^{1/2}}{z^2}\, dz \tag{1.23}$$

$$\gtrsim 2e^R \int_2^{e^W} \frac{1}{z^2}\, dz \tag{1.24}$$

$$\gtrsim e^R \tag{1.25}$$

so

$$\frac{A_{-N-1}}{A_{-N}} \gtrsim \frac{1}{4}e^{W/2}. \tag{1.26}$$

Finally we note that $\frac{1}{4}e^{W/2}$ can be made as large as desired, in particular larger than 2, which completes the argument that the relative densities of the set S of black stripes does not have a well defined limit.

The example of the stripe model in the hyperbolic plane, where the stripes are all of equal "width", is more unsettling than the example of annuli in Euclidean space discussed above, where in a sense the oscillation was more obviously built in. We will see below that this stripe model is only a simple version of a well known disk packing.

There has been another common way to compute or estimate the density of packings in Euclidean spaces, using tilings associated with the packings, and the relative densities of the bodies in the tiles. (A tile is a homeomorphic image of the closed unit ball, and a tiling is a packing by tiles for which the union of the tiles is the full space $\mathbb{S}$.) We emphasize that this is an attempt to reduce the intuitive global idea of density, which involves taking a limit of approximate densities in expanding regions of finite volume, to a more local notion. As a significant example of this approach we note an elegant proof [27], [25] of the optimal density for packings of equal disks in the Euclidean plane. The proof uses the Vorononi cells of the bodies of a packing, where the cell for a body β is the set of all points $p \in \mathbb{S}$ as close to β as to any body of the packing. The proof shows that the relative density in its Voronoi cell of any disk of any packing is bounded above by that of any of the Voronoi cells in the obvious hexagonal packing. This argument was extended to sphere packings in $\mathbb{S}$ by K. Böröczky, who showed [6] that the relative density of any sphere of any packing of $\mathbb{S}$ in its Voronoi cell is bounded above by the relative density associated with that of a regular simplex. (See [26] for details.) Such relative densities in tiles of associated tilings have remained an important tool in analyzing optimal densities of sphere packings in Euclidean spaces [26], [3].

1.2.3 tight radius packings

In hyperbolic space, particularly the plane $\mathbb{H}^2$, the above method of estimating or computing a density of sphere packings through an associated tiling has been used convincingly for the special case of disks of "tight" radius. The radius r of a sphere in $\mathbb{S}$ is called tight if the regular simplex of side length $2r$ admits a (full-face to full-face) tiling of $\mathbb{S}$. In $\mathbb{H}^2$ this is the case if and only if the equilateral triangle of edge length $2r$ has angles of the form $2\pi/n$ for some $m \geq 7$, in which case

$$2r = 2r_m = \cosh^{-1}[\cot(\frac{\pi}{m})\cot(\frac{2\pi}{m})]. \tag{1.27}$$

and clearly $r_m \to \infty$ as $m \to \infty$. For disks with tight radius r_m the obvious "periodic" packing, in which each disk is surrounded by m disks touching it, has a well defined density in the sense that, besides the method using Voronoi tilings, *any* reasonable way to compute the density would give the same value (namely $[3\csc(\pi/m) - 6]/[m-6]$), in particular any limit of the form (1.8) [9].

1.2.4 Böröczky's packing

There is an influential example due to Böröczky [5] which points out a difficulty in using relative density in tiles to define the density of at least some packings in hyperbolic space, even some which are rather symmetric. Place disks in the upper half plane model of the hyperbolic plane with Euclidean centers at those points with coordinates

$$\{(e^{2j+\frac{1}{2}}(k+1/2),\ e^{2j+\frac{1}{2}})\,|\,j,k\in Z\}. \tag{1.28}$$

The connection between this and the stripe model is simple: we are placing the disks equally spaced in the black stripes (and we are taking the value $W = 1$ for the width of the stripes). See Figure 1.1 for a picture of the packing, which includes some horocycles and geodesics to help understand the structure.

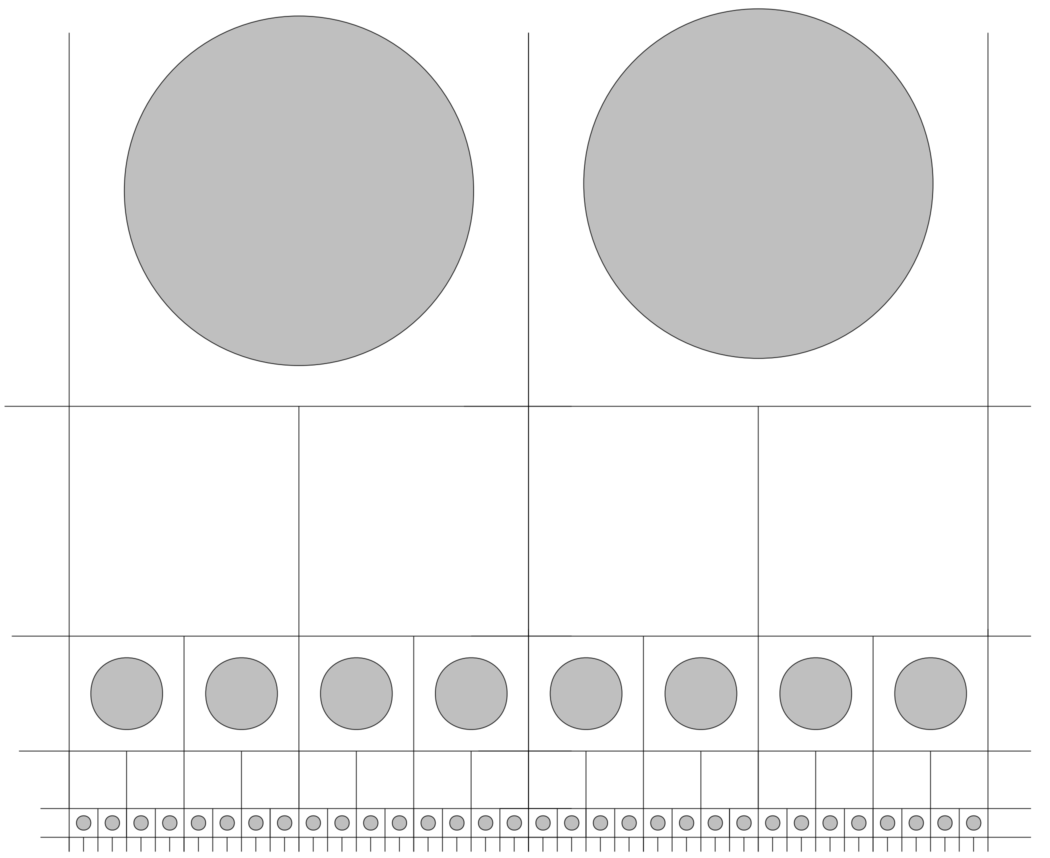

Figure 1.1: Boroczky's packing of disks

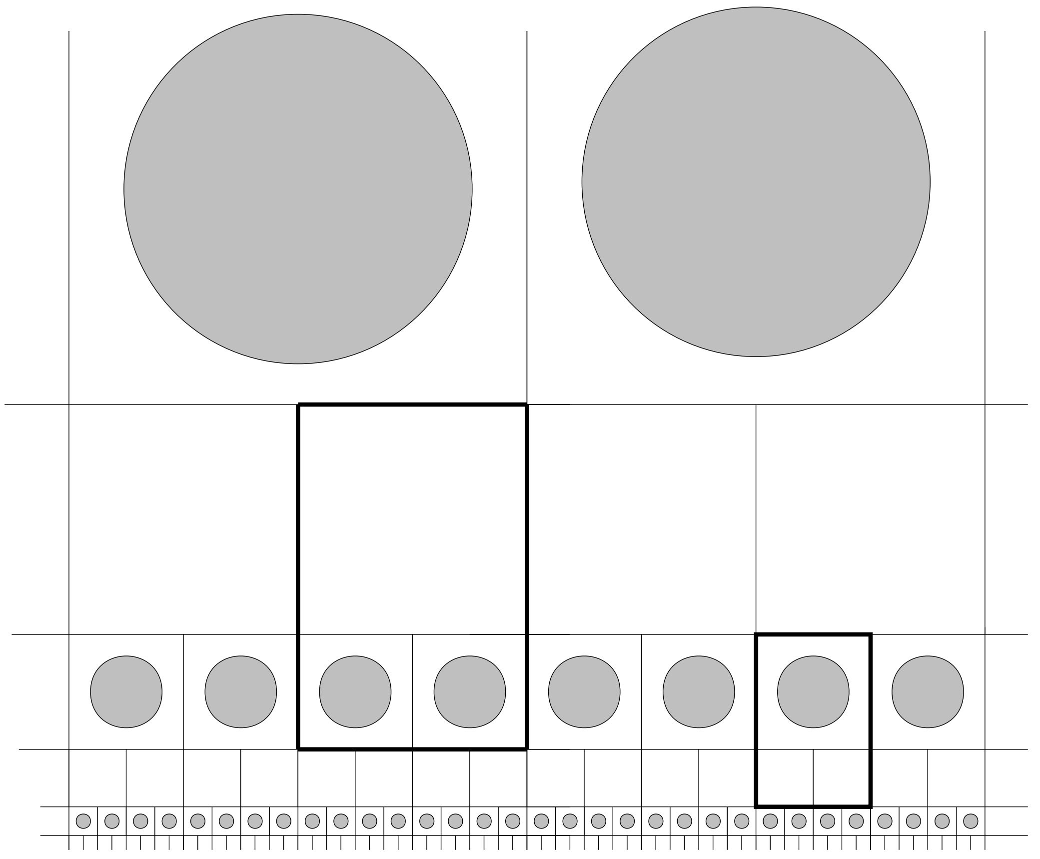

Figure 1.2: Boroczky's packing of disks with two tiles in dark outline

In Figure 1.2 we see the same packing with two congruent tiles in dark outline. For each tile consider the tiling of the plane made by congruent copies of the tile, as follows. First produce copies of the tile by the congruences: $(x, y) \to (x + mw, y)$, $m \in Z$, where w is the Euclidean width of the body. This fills out a black and white stripe. Then produce, from these, more copies of the tile by the congruences: $(x, y) \to (e^{2m}x, e^{2m}y)$, $m \in Z$. Together these copies of the original tile will cover the whole plane. The two tilings made this way, one from each of the tiles in Figure 1.2, are both simply related to the same packing of disks. The punchline is, the tiling made by starting with the tile on the left in Figure 1.2 would suggest assigning a "density" of the packing of disks twice the value suggested by the tiling made by starting with the tile on the right! We repeat the point that using a tiling to compute the density of some packing, thus making the computation more local, is useful in Euclidean spaces but is less convincing in hyperbolic spaces.

We now return to the question of a definition of optimally dense packings of $\mathbb{H}^n$. As we say above, for packings of Euclidean space the notion of densest packings is easy to clarify, and one way to understand this is through the computation of the ratio $f(\rho, a)$ of volumes of concentric spheres of radii ρ and $\rho + a$.

Note that:

i) in $\mathbb{E}^n$ $f(\rho, a) \equiv \dfrac{\rho^n}{(\rho + a)^n}$, so for fixed $a > 0$ and n, $f(\rho, a) \to 1$ as $\rho \to \infty$;

ii) in $\mathbb{E}^n$, for fixed $a > 0$ and ρ, $f(\rho, a) \to 0$ as $n \to \infty$;

iii) in $\mathbb{H}^n$, for fixed $a > 0$ and n, $f(\rho, a) \to e^{-ca}$ as $\rho \to \infty$, for some constant $c > 0$.

To see why these phenomena interfere with a generalization to hyperbolic space of the method used earlier for Euclidean packings, consider the packings $\mathcal{P}_\rho$ of the hyperbolic plane, by disks of fixed radius R, defined for each $\rho >> 0$ as follows. For each sufficiently large radius $\rho >> R$, place disks of radius R on the circumference of a circle C_ρ of radius ρ, so that: they cover all but perhaps one arc of the circumference; there are as many disks as possible without overlap; disks intersect only at points of the circumference. We now show that by taking R (and therefore ρ) large enough we can ensure that the fraction of the area of C_ρ covered by the disks is as close to 1 as desired.

The fraction of the area of C_ρ which is in the annulus between C_ρ and the concentric $C_{\rho'}$ for $\rho > \rho'$ is of the order $1 - e^{\rho' - \rho}$ for large $\rho,\ \rho'$, and by taking $0 << \rho - \rho' << R << \rho' << \rho$ we can ensure that most of this area is inside the disks of radius R – all except those regions outside pairs of touching disks of radius R and outside the circle $C_{\rho'}$, plus the region near any uncovered arc of C_ρ.

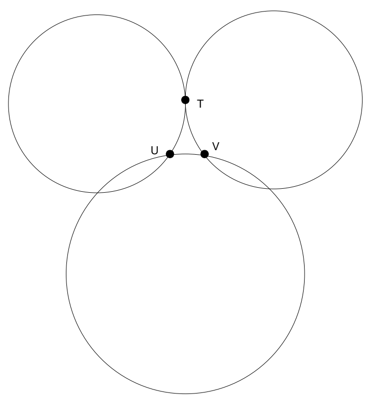

Figure 1.3: Uncovered regions

But using the convexity of circles, the former regions are each contained in triangles of the form TUV (see Figure 1.3), so have negligible area, and another simple triangle argument applies to the region near any uncovered arc of C_ρ.

So by choosing R appropriately we could get almost all the area of C_ρ to lie outside C'_ρ.

Where in the Euclidean argument we used larger and larger cubes, in hyperbolic space we would use fundamental domains of cocompact subgroups of the isometry group $\mathcal{G}$ of $\mathbb{H}^n$. But we needed the fact, in Euclidean space, that the volume of the portion of a packing near the boundary of the fundamental domain would be negligible, while we see now that for large fundamental domains and large bodies, this is far from the case. In summary, where we used i) to show the existence of optimal packings in Euclidean space, in hyperbolic space we have instead iii), which for large spheres is approximately ii). This is the intuitive reason why there has been difficulty defining optimally dense packings in hyperbolic space for so long.

Chapter 2

Invariant Measures

We now discuss an approach to density specifically aimed at controlling those packings, such as the above example of Böröczky, which pose difficulty in computing a reliable density. Even though the methods are also applicable to Euclidean space, the interests of this article make it natural to specialize the discussion from now on to $\mathbb{S} = \mathbb{H}^n$.

The key idea is to use a pointwise ergodic theorem of Nevo ([19],theorem 1) for dimension $n \geq 3$; ([20] Thm. 3) for $n \geq 2$), the conclusion of which is the existence of limits of the type (1.8) in the intuitive definition of density. The fact that such theorems only prove existence of the limit "almost everywhere" is not a defect, it is a feature, necessitated by examples such as that of Böröczky.

We begin by reproducing some notation and results from [9]. Let $d(\cdot,\cdot)$ be the usual metric on $\mathbb{S}$, and let $\mathcal{O}$ be a distinguished origin. We suppose given a finite collection $\mathcal{B}$ of bodies β_j in $\mathbb{S}$. Let $\Sigma_{\mathcal{B}}$ be the space of all "saturated" packings of $\mathbb{S}$ by congruent copies of the β_j, that is, packings $\mathcal{P}$ with the property that any congruent copy of a body in $\mathcal{B}$ intersects a body of $\mathcal{P}$. On $\Sigma_{\mathcal{B}}$ we put the following metric, corresponding to uniform convergence on compact subsets of $\mathbb{S}$:

$$d_{\mathcal{B}}(\mathcal{P}_1, \mathcal{P}_2) = \sup_{k \geq 1} \frac{1}{k} h(B_k \cap \mathcal{P}_1, B_k \cap \mathcal{P}_2), \tag{2.1}$$

where B_k denotes the closed ball of radius k centered at the origin, and for compact sets A and C we use the Hausdorff metric

$$h(A, C) \equiv \max\{\sup_{a \in A} \inf_{c \in C} d(a, c), \sup_{c \in C} \inf_{a \in A} d(a, c)\}. \tag{2.2}$$

It is not hard to see [23] that $\Sigma_{\mathcal{B}}$ is compact in this metric topology, and that the natural action: $(g, \mathcal{P}) \in \mathcal{G} \times \Sigma_{\mathcal{B}} \longrightarrow g(\mathcal{P}) \in \Sigma_{\mathcal{B}}$ of the isometry group $\mathcal{G}$ of $\mathbb{S}$ on $\Sigma_{\mathcal{B}}$ is (jointly) continuous. Let $\mathcal{M}(\mathcal{B})$ be the family of Borel probability measures on $\Sigma_{\mathcal{B}}$. We call a measure $\mu \in \mathcal{M}(\mathcal{B})$ "invariant" if for any Borel subset $E \subset \Sigma_{\mathcal{B}}$ and any $g \in \mathcal{G}$, $\mu(gE) = \mu(E)$. Let $\mathcal{M}_I(\mathcal{B})$ be the subset of invariant measures and $\mathcal{M}_I^e(\mathcal{B})$ the convex extreme ("ergodic") points of $\mathcal{M}_I(\mathcal{B})$, all in their weak* topology, in which $\mathcal{M}(\mathcal{B})$ and $\mathcal{M}_I(\mathcal{B})$ are compact.

We will study these ergodic measures as a substitute for studying individual packings. As we will see, for any ergodic measure $\mu \in \mathcal{M}_I(\mathcal{B})$ there is a set of packings Z of full μ-measure such that for each $\mathcal{P} \in Z$, the orbit of $\mathcal{P}$ is dense in the support of μ. So studying μ is a lot like studying a packing in Z. We will make this relationship more clear in what follows but first some examples.

Suppose $\mathcal{P}$ is a "periodic" packing, i.e. the symmetry group $\Gamma_{\mathcal{P}}$ of $\mathcal{P}$ is cocompact in $\mathcal{G}$. We will construct a measure $\mu_{\mathcal{P}} \in \mathcal{M}_I^e(\mathcal{B})$ whose support is contained in the orbit $O(\mathcal{P}) \equiv \{g\mathcal{P} \,|\, g \in \mathcal{G}\} \subset \Sigma_{\mathcal{B}}$ of $\mathcal{P}$. $O(\mathcal{P})$ is naturally homeomorphic to the (metrizable) space $\mathcal{G}/\Gamma_{\mathcal{P}}$ of left cosets by the homeomorphism $q_{\mathcal{P}} : O(\mathcal{P}) \to \mathcal{G}/\Gamma_{\mathcal{P}}$ with $q_{\mathcal{P}}(g\mathcal{P}) = g\Gamma_{\mathcal{P}}$. There is a natural probability measure on $\mathcal{G}/\Gamma_{\mathcal{P}}$ induced by Haar measure on $\mathcal{G}$ by the projection map $\pi_{\mathcal{P}} : \mathcal{G} \to \mathcal{G}/\Gamma_{\mathcal{P}}$. (Aside from an overall normalization the measure on $\mathcal{G}/\Gamma_{\mathcal{P}}$ can be defined on sufficiently small open balls $B \subset \mathcal{G}/\Gamma_{\mathcal{P}}$ as the Haar measure of any of the components of $\pi_{\mathcal{P}}^{-1}(B)$.) Hence $q_{\mathcal{P}}$ induces a probability measure $\hat{\mu}_{\mathcal{P}}$ on $O(\mathcal{P})$. This measure can then be extended to all of $\Sigma_{\mathcal{B}}$ in the following way: $\mu_{\mathcal{P}}(E) = \hat{\mu}_{\mathcal{P}}(E \cap O(\mathcal{P}))$ for any Borel set $E \subseteq \Sigma_{\mathcal{B}}$. We will use the term "periodic measure" to denote any measure in $M_I(\mathcal{B})$ associated in this way with the orbit of a periodic packing. It is not hard to prove from the uniqueness of Haar measure on $\mathcal{G}$ that there is only one probability measure, with support in the orbit of a periodic packing, which is invariant under $\mathcal{G}$.

Next, we define the density of an invariant measure. After the definition, we will show how the density of an invariant measure relates to the density of packings in its support.

For any $p \in \mathbb{S}$ we define the real valued function F_p on $\Sigma_{\mathcal{B}}$ as the indicator function of the set of all packings $\mathcal{P}$ such that p is contained in a body of $\mathcal{P}$. (The latter condition will sometimes be expressed as $p \in \mathcal{P}$.)

Definition 1. For any invariant measure $\mu \in \mathcal{M}_I(\mathcal{B})$, the "average density" $D(\mu)$ is defined as $\int_{\Sigma_{\mathcal{B}}} F_p(y)\, d\mu(y)$.

Note: the average density $D(\mu)$ is independent of the choice of p, because of the invariance of the measure, so p is not needed in the notation. For convenience we sometimes use $p = \mathcal{O}$.

If $\mathcal{P}_\mu$ is a random packing with distribution μ then the above definition states that the density of μ is the probability that the origin is contained in a body of $\mathcal{P}_\mu$.

For periodic packings $\mathcal{P}$ there is an obvious notion of density using a fundamental domain of $\Gamma_{\mathcal{P}}$. The above definition of density coincides with this intuitive notion for such special $\mathcal{P}$.

Proposition 2.0.2. *If $\mathcal{P}$ is a periodic packing, $D(\mu_x)$ is the relative volume of any fundamental domain for Γ_x taken up by the bodies of x.*

We will prove this proposition in the next section.

2.1 Why Invariant Measures are good

We need the following notation. As usual we let $\mathcal{G}$ denote the group of orientation preserving isometries of hyperbolic n-space $\mathbb{H}^n$ (for some fixed $n \geq 2$). Let $\pi : \mathcal{G} \to \mathbb{H}^n$ be the projection map $g \to g\mathcal{O}$ where $\mathcal{O}$ is some fixed point in $\mathbb{H}^n$. Then we let $\tilde{B}_k$ denote the inverse image under π of the closed ball of radius k centered at $\mathcal{O}$. Finally let $\lambda_{\mathcal{G}}$ denote a Haar measure on $\mathcal{G}$, normalized so that $\lambda_{\mathcal{G}}(\tilde{B}_k)$ is the volume of the k-ball in $\mathbb{H}^n$.

We will use the following special case of Theorem 3 in [20] to relate the density of an ergodic measure to the density of (almost every) packing in its support.

Theorem 2.1.1 (Nevo). *If $\mathcal{G}$ acts continuously on a compact metric space X such that there is a Borel probability measure μ on X that is invariant and ergodic under this action, then for every function $f \in L^p(X, \mu)$ $(1 < p < \infty)$ there is a set Z of full μ measure such that for every $z \in Z$,*

$$\int_X f d\mu = \lim_{n\to\infty} \frac{1}{\lambda_{\mathcal{G}}(\tilde{B}_n)} \int_{\tilde{B}_n} f(gz)\, d\lambda_{\mathcal{G}}(g). \tag{2.3}$$

Actually we will use the following extension of this result.

Theorem 2.1.2. *Under the same hypotheses as the above theorem, the set Z may be taken to be invariant under G.*

We will prove this result in the next section. Applying Theorem 2.1.2 to the function F_p we get

Theorem 2.1.3. *If $\mu \in M_I^e(\mathcal{B})$ then there exists a set Z of full μ-measure such that for all $p \in \mathbb{H}^n$ and all $\mathcal{P} \in Z$,*

$$\lim_{r\to\infty} \frac{vol[\mathcal{P} \cap B_p(r)]}{vol[B_p(r)]} = D(\mu). \tag{2.4}$$

Note that this implies that the orbit of the stripe model has measure zero with respect to every invariant measure. We will give another explanation for this fact in a later section.

From example 1.2.4 we concluded that it is not possible, in general, to compute the density of a hyperbolic packing using a tiling associated to it. In spite of this we will show that it is possible to compute the density of an invariant measure using an associated space a tilings.

Let Σ be a (compact, invariant) space of packings of $\mathbb{H}^n$. Let μ be a $Isom^+(\mathbb{H}^n)$ invariant measure on Σ. Suppose that Θ is a space of tilings (of $\mathbb{H}^n$) and that there is an equivariant map $\phi : \Sigma \to \Theta$. For example, Θ may be the space of Voronoi (or Delone) tilings corresponding to Σ. For $\mathcal{P} \in \Sigma$ such that the origin is contained in a tile of $\phi(\mathcal{P})$, let $\tau_p(\mathcal{P})$ denote the tile of $\phi(\mathcal{P})$ containing the point $p \in \mathbb{H}^n$. We claim that for any p:

$$D(\mu) = \int_\Sigma \frac{vol[\mathcal{P} \cap \tau_p(\mathcal{P})]}{vol[\tau_p(\mathcal{P})]} \, d\mu(\mathcal{P}). \tag{2.5}$$

Define a function $f : \mathbb{H}^n \times \mathbb{H}^n \times \Sigma \to \mathcal{R}$ by $f(p, q, \mathcal{P}) = 1/vol[\tau_p(\mathcal{P})]$ if p and q are in the same Voronoi cell of $\mathcal{P}$ and p is in a body of $\mathcal{P}$ (otherwise $f(p, q, \mathcal{P}) = 0$). Define a measure ν on $\mathbb{H}^n \times \mathbb{H}^n$ by

$$\nu(E \times F) = \int_\Sigma \int_E \int_F f(p, q, \mathcal{P}) \, dvol(p) \, dvol(q) \, d\mu(\mathcal{P}). \tag{2.6}$$

Since μ is invariant, it is easy to check that for all $g \in Isom^+(\mathbb{H}^n)$, $\nu(gE \times gF) = \nu(E \times F)$. The mass-transport principle [2] implies that $\nu(E \times \mathbb{H}^n) = \nu(\mathbb{H}^n \times E)$ for any measureable $E \subset \mathbb{H}^n$. But it can easily be checked that $\nu(E \times \mathbb{H}^n) = vol(E)D(\mu)$ and

$$\nu(\mathbb{H}^n \times E) = vol(E) \int_\Sigma vol(\mathcal{P} \cap \tau_p(\mathcal{P}))/vol(\tau_p(\mathcal{P})) \, d\mu(\mathcal{P}) \tag{2.7}$$

(for any p). This proves the claim. For emphasis, we repeat that when μ is an invariant measure we can compute its density with respect to local structures such as the Voronoi or Delone tilings. If $\mu_{\mathcal{P}}$ is a periodic measure and Θ is the space of tilings by a fundamental domain of $\Gamma_{\mathcal{P}}$ then there is a natural equivariant map from the orbit of $\mathcal{P}$ to Θ. The above result then yields Proposition 2.0.2.

The proof of theorem 2.1.2 will follow from the preparation of the next few lemmas.

We denote by $vol(E)$ the volume of a measureable subset E of hyperbolic n-space. For any point $p \in \mathbb{H}^n$ and $R > 0$, we let $B_R(p)$ denote the ball of radius R centered at p. If $p = \mathcal{O}$, we sometimes write B_R instead of $B_R(\mathcal{O})$. We let $S_R(p)$ denote the sphere of radius R centered at p.

Lemma 2.1.4. *If the distance between points p and q in $\mathbb{H}^n$ is r then*

$$F(r) := \lim_{R\to\infty} \frac{vol(B_R(p) - B_R(q))}{vol(B_R)} \tag{2.8}$$

exists and $\lim_{r\searrow 0} F(r) = 0$*.*

Proof. Let r be fixed for now. Let M_R (for middle) denote the convex hull of p, q and $S_R(p) \cap S_R(q)$. For example, in 2-dimensions, M_R is a quadrilateral. Let $O_R(p)$ (for obtuse cone) denote the component of $B_R(p) - M_R$ whose closure contains p. Similarly let $O_R(q)$ denote the component of $B_R(q) - M_R$ whose closure contains q. Let $A_R(q)$ (for acute cone) denote $B_R(q) - O_R(q)$. Since $B_R(p) - B_R(q) = O_R(p) - A_R(q)$ and $A_R(q) - O_R(p) = M_R$, we have $vol(B_R(p) - B_R(q)) = vol(O_R(p)) - vol(A_R(q)) + vol(M_R)$. Thus it suffices to show that

$$\lim_{R\to\infty} \frac{vol(O_R(p)) - vol(A_R(q))}{vol(B_R)} \tag{2.9}$$

exists and goes to 0 as $r \to 0$ and

$$\lim_{R\to\infty} \frac{vol(M_R)}{vol(B_R)} = 0. \tag{2.10}$$

We prove second claim first. Let H be two dimensional geodesic plane containing p and q. Let $x_R \in H \cap B_R(p) \cap B_R(q)$. Note (from two-dimensional geometry) that the angle $\angle p x_R q$ goes to 0 as $R \to 0$. In fact we can compute from the law of cosines,

$$\cos(\angle p x_R q) = \frac{\cosh^2(R) - \cosh(r)}{\sinh^2(R)} \to 1 \text{ as } R \to \infty. \tag{2.11}$$

Let $W >> 0$. Let $\{y_R, z_R\} = H \cap \partial M_R \cap B_p(R - W)$. Let g_R be a hyperbolic isometry taking H to the upper-half plane $\{(x_1, 0, 0, .., 0, x_n) | \, x_1 \in \mathbb{R}, x_n > 0\}$ (in the upper-half space model) and such that $y_R = (0, 0, ..., 0, 1)$ and $x_R = (0, 0, .., e^W)$. Then the nth coordinate of $g_R z_R$ goes to 1 as $R \to \infty$ because of the general principle that spheres limit onto horospheres. However, the angle $\angle y_R x_R z_R$ equals the angle $\angle p x_R q$ and so goes to 0 as $R \to \infty$. Hence the distance between z_R and y_R goes to 0 as $R \to \infty$. Now it should be clear that the area of the triangle $x_R y_R z_R$ goes to zero as R goes to infinity. Note that this triangle contains $H \cap M_R \cap [B_R(p) - B_{R-W}(p)]$.

We wish to show that the relative fraction of volume of $B_p(R)$ taken up by $M_R \cap [B_R(p) - B_{R-W}(p)]$ goes to zero as $R \to \infty$. Note that $M_R \cap [B_R(p) - B_{R-W}(p)]$ is obtained by revolving $H \cap M_R \cap [B_R(p) - B_{R-W}(p)]$ about the geodesic γ that passes through p and q. Thus it suffices to show that if T_R is any region in $H \cap B_R(p)$ such that the area of T_R goes to 0 as $R \to \infty$ then the relative fraction of volume in $B_R(p)$ taken up by the body obtained by revolving T_R about γ goes to zero as $R \to \infty$. For this, consider the "worst case scenario" when T_R is as far as possible from γ (and thus the body of revolution has as much as volume as possible). Then there exists $\epsilon_R > 0$ such that $\epsilon_R \to 0$ as $R \to \infty$ and T_R is contained in $H \cap [B_R(p) - B_{R-\epsilon_R}(p)]$. But then the body obtained from revolving T_R about γ is contained in $B_R(p) - B_{R-\epsilon_R}(p)$. By theorem 1.1.1

$$\frac{vol[B_R(p) - B_{R-\epsilon_R}(p)]}{vol(B_R(p)} \to 1 - e^{-c_1\epsilon_R} \to 0 \tag{2.12}$$

as $R \to \infty$. We have now shown

$$\lim_{R\to\infty} \frac{vol(M_R \cap [B_R(p) - B_{R-W}(p)]}{vol(B_R(p))} = 0. \tag{2.13}$$

Since

$$\lim_{R\to\infty} \frac{vol(B_{R-W}(p))}{vol(B_R(p))} = e^{-c_1W}, \tag{2.14}$$

$$\lim_{R\to\infty} \frac{vol(M_R)}{vol(B_R(p))} = \lim_{R\to\infty} \frac{vol(M_R \cap B_{R-W}(p))}{vol(B_R(p))} \leq e^{-c_1W}. \tag{2.15}$$

Since W is arbitrary, $\frac{vol(M_R)}{vol(B_R(p))} \to 0$ as $R \to \infty$ as claimed.

To see the first claim, let K be the half space containing q whose boundary is the perpendicular bisector of the seqment with endpoints p and q. Let Q be the convex hull of p and K. Then

$$\lim_{R\to\infty} \frac{vol(O_p(R))}{vol(B_R(p))} = \frac{vol(B_R(p) - Q)}{vol(B_R(p))} \tag{2.16}$$

where the R on the right hand side is arbitrary. In fact this limit converges montonically. Thus, as $r \to \infty$, the above limit equals $1/2$. Since $vol(A_R(q)) = vol(B_R(p)) - vol(O_R(p))$, this finishes the lemma.

$\square$

For $h : \mathcal{G} \to \mathbb{R}$ Borel let $A_-(h), A_+(h) : \mathcal{G} \to \mathbb{R} \cup \{\pm\infty\}$ be defined by

$$A_-(h)(g) = \lim\inf_{R\to\infty} \frac{1}{vol(B_R)} \int_{\tilde{B}_R} h(g'g)\, d\lambda_{\mathcal{G}}(g') \tag{2.17}$$

$$A_+(h)(g) = \limsup_{R\to\infty} \frac{1}{vol(B_R)} \int_{\tilde{B}_R} h(g'g)\, d\lambda_{\mathcal{G}}(g'). \tag{2.18}$$

Corollary 2.1.5. *If h is any Borel function on G then $A_+(h)$ and $A_-(h)$ are continuous.*

Proof. Let $g_1, g_2 \in \mathcal{G}$. Let $r = d(g_1\mathcal{O}, g_2\mathcal{O})$. Then

$$\begin{aligned}
A_+(h)(g_1) &= \limsup_{R\to\infty} \frac{1}{vol(B_R)} \int_{\tilde{B}_R} h(g'g_1)\, d\lambda_{\mathcal{G}}(g') && (2.19)\\
&= \limsup_{R\to\infty} \frac{1}{vol(B_R)} \int_{\tilde{B}_R g_1^{-1}} h(g')\, d\lambda_{\mathcal{G}}(g') && (2.20)\\
&= \limsup_{R\to\infty} \frac{1}{vol(B_R)} \int_{\pi^{-1}(B_R(g_1\mathcal{O}))} h(g')\, d\lambda_{\mathcal{G}}(g') && (2.21)\\
&\geq \limsup_{R\to\infty} \frac{1}{vol(B_R)} \int_{\pi^{-1}(B_R(g_1\mathcal{O}))\cap\pi^{-1}(B_R(g_2\mathcal{O}))} h(g')\, d\lambda_{\mathcal{G}} && (2.22)\\
&\geq \limsup_{R\to\infty} \frac{1}{vol(B_R)} \Big(\int_{\pi^{-1}[B_R(g_2\mathcal{O})]} h(g')\, d\lambda_{\mathcal{G}}(g') && (2.23)\\
&- \int_{\pi^{-1}[B_R(g_2\mathcal{O})]-\pi^{-1}[B_R(g_1\mathcal{O})]} h(g')\, d\lambda_{\mathcal{G}}(g') \Big) && (2.24)\\
&\geq A_+(h)(g_2) && (2.25)\\
&- \limsup_{R\to\infty} \frac{1}{vol(B_R)} \int_{\pi^{-1}[B_R(g_2\mathcal{O})]-\pi^{-1}[B_R(g_1\mathcal{O})]} h(g')\, d\lambda_{\mathcal{G}} && (2.26)\\
&\geq A_+(h)(g_2)\Big(1 - \limsup_{R\to\infty} \frac{vol[B_R(g_2\mathcal{O}) - B_R(g_1\mathcal{O})]}{vol(B_R)}\Big) && (2.27)\\
&= A_+(h)(g_2)(1 - F(r)). && (2.28)
\end{aligned}$$

Since g_1 and g_2 are arbitrary, $A_+(h)$ is continuous. The proof for $A_-(h)$ is similar. ◻

Proof. (of theorem 2.1.2): Let G_0 be countable dense subset of G. Let Z_0 be as in theorem 1.1. Let $Z = \bigcap_{g\in G_0} g^{-1}Z_0$. Since Z is a countable intersection of sets of measure 1, $\mu(Z) = 1$. By definition, for all $z \in Z$ and for all $g \in G_0$,

$gz \in Z_0$. For $x \in X$, define $h_x : G \to \mathbb{R}$ by $h_x(g) = f(gx)$. For $z \in Z$ and $g \in G_0$, we have

$$A_+(h_z)(g) = \limsup_{R\to\infty} \frac{1}{vol(B_R)} \int_{\tilde{B}_R} h_z(g'g)\, d\lambda_G(g') \quad (2.29)$$

$$= \limsup_{R\to\infty} \frac{1}{vol(B_R)} \int_{\tilde{B}_R} f(g'gz)\, d\lambda_G(g') \quad (2.30)$$

$$= \int_X f d\mu. \quad (2.31)$$

The last equation holds since $gz \in Z_0$. Since $A_+(h_z)$ is continuous, we have that the above equations hold for all $g \in G$. Similarly, we get that $A_-(h_z)(g) = \int_X f d\mu$ for all $g \in G$ and $z \in Z$. Now we are done. □

2.2 Optimally dense measures and packings

We now define optimality through measures.

Definition 2. $D(\mathcal{B}) \equiv \sup_{\mu \in \mathcal{M}_I^e(\mathcal{B})} D(\mu)$ will be called the "optimal density for $\mathcal{B}$", and any ergodic measure $\tilde{\mu} \in \mathcal{M}_I^e(\mathcal{B})$ will be called "optimally dense (for $\mathcal{B}$)" if $D(\tilde{\mu}) = D(\mathcal{B})$. We define "optimally dense packings" a little differently than in [9]. We say that a packing $\mathcal{P}$ is optimally dense if there is an optimally dense measure μ such that the orbit of $\mathcal{P}$ is dense in the support of μ and for every $p \in \mathbb{H}^n$ $D(\mu)$ is equal to the limit of the relative fraction of volume in expanding sphere centered at p taken up by bodies of $\mathcal{P}$.

Theorem 2.2.1. *For any finite collection $\mathcal{B}$ of bodies there exists an optimally dense measure μ on $\Sigma_{\mathcal{B}}$.*

Note: There may be many optimally dense measures for a given $\mathcal{B}$.

Proof. We let $\mathcal{M}_I$ have the weak* topology. Equivalently $\mu_n \to \mu \in \mathcal{M}_I$ if and only if for every continuous function $f : \Sigma_{\mathcal{B}} \to \mathbb{R}$, $\int_{\Sigma_{\mathcal{B}}} f\, d\mu_n \to \int_{\Sigma_{\mathcal{B}}} f\, d\mu$. By standard functional analysis, since $\Sigma_{\mathcal{B}}$ is a compact metric space, $\mathcal{M}_I$ is compact.

Let $J = \{\mathcal{P} \in \Sigma_{\mathcal{B}} | \, \mathcal{O} \in \mathcal{P}\}$. Let χ_J be the characteristic function of J. Because χ_J is upper semicontinuous there exists a decreasing sequence f_j of continuous real valued functions on $\Sigma_{\mathcal{B}}$ which converge pointwise to χ_J. Choose a sequence $\mu_k \in \mathcal{M}_I$ such that $D(\mu_k) = \int_{\Sigma_{\mathcal{B}}} F_{\mathcal{O}}\, d\mu_k \to D(\mathcal{B})$ as $k \to \infty$, and, using the compactness of $\mathcal{M}_I$, assume without loss of generality that μ_k converges to some $\mu_\infty \in \mathcal{M}_I$. Then $\int_{\Sigma_{\mathcal{B}}} f_j\, d\mu_k \to \int_{\Sigma_{\mathcal{B}}} f_j\, d\mu_\infty$ as $j \to \infty$, and $\int_{\Sigma_{\mathcal{B}}} f_j\, d\mu_\infty \searrow D(\mu_\infty)$ as $k \to \infty$. Since $\int_{\Sigma_{\mathcal{B}}} f_j\, d\mu_k \geq D(\mu_k)$ and $D(\mu_k) \to D(\mathcal{B})$ as $k \to \infty$, $D(\mu_\infty) \geq D(K)$. From the Krein-Milman theorem there exists an ergodic measure $\tilde{\mu} \in \mathcal{M}_I$ for which $D(\tilde{\mu}) = \int_{\Sigma_{\mathcal{B}}} \chi_J\, d\tilde{\mu} \geq D(\mu_\infty)$, and thus $D(\tilde{\mu}) \geq D(\mathcal{B})$. But then from the definition of $D(\mathcal{B})$, $D(\tilde{\mu}) = D(\mathcal{B})$. $\square$

Theorem 2.2.2. *For any optimally dense measure μ, there exists a set Z of optimally dense packings such that $\mu(Z) = 1$.*

Proof. Choose a countable dense subset A in the support $supp(\mu)$ of μ. Let C be the collection of all balls whose center lies in A and whose radius is $1/n$ for some positive integer n. For each of the balls $c \in C$ there is a subset of $supp(\mu)$, of full μ-measure, whose orbit intersects c, as we see by applying the ergodic theorem to the indicator function of c. It follows that there is a set of full μ-measure of points each of whose orbit intersects every ball in C. The closure of any such orbit must therefore contain A. Since A is dense in $\sigma(\tilde{\mu})$, any such orbit must also be dense. Thus there is a set $Z_0 \subset supp(\mu)$ of full measure such that the orbit of each $\mathcal{P} \in Z_0$ is dense in $supp(\mu)$. Clearly Z_0 can be chosen to be invariant. Let Z_1 be the set given by theorem 2.1.2. Then $Z_0 \cap Z_1$ is a full μ-measure set of optimally dense packings.

□

2.2.1 Globally densest measures are locally densest

A packing is called completely saturated [12] if it is not possible to replace a finite number of bodies of the packing with a greater total volume of bodies and still remain a packing. Intuitively, we think of a completely saturated packing as one that is locally densest. In [12], it was proven that any convex body of Euclidean space admits a completely saturated packing (and more generally any body with the strict nested similarity property) (see also [16]).

Theorem 2.2.3. *For any optimally measure μ, the set of completely saturated packings has μ-measure 1. In other words, if $\mathcal{P}_\mu$ is a random packing with optimally dense distribution μ then $\mathcal{P}_\mu$ is completely saturated almost surely.*

By a fundamental domain F of a subgroup $\mathcal{G}' < \mathcal{G}$, we shall mean a connected set in $\mathbb{S}$ equal to the closure of its interior such that $\{gF |\, g \in \mathcal{G}'\}$ is a packing by F and $\bigcup_{g \in \mathcal{G}'} gF = \mathbb{S}$.

Let $\{\mathcal{G}_j\}_{j=0}^{\infty}$ be a sequence of discrete cocompact subgroups of $\mathcal{G}$ such that there exist fundamental domains F_j (in $\mathbb{S}$) for $\mathcal{G}_j$ with $B_j \subset F_j$. We will also assume that F_j and $\mathcal{G}_j$ have been chosen so that for all $g \in \pi^{-1}(F_j)$, $g^{-1} \in \pi^{-1}(F_j)$. For the Euclidean case, we could, for example, let $\mathcal{G}_j$ be the cubic lattice generated by the translations $\tau_{i,j} := (x_1, ..., x_n) \to (x_1, ..., x_i + j, x_{i+1}, ..., x_n)$ for $1 \le i \le n$. Then we could choose F_j to be the cube of side length j whose center is the origin and whose faces are parallel to the coordinate planes. For the hyperbolic case, we refer to ([12], Theorem 4.1) for the existence of $\{\mathcal{G}_j\}_{j=0}^{\infty}$.

For $\mathcal{P} \in \Sigma_{\mathcal{B}}$ and $F \subset \mathbb{S}$, let $\mathcal{P} * F$ be the packing consisting of all elements $gK \in \mathcal{P}$ such that the interior of gK intersects F nontrivially. Also let ∂F denote the boundary of F, i.e. the intersection of the closure of F with the closure of its complement. Let $X_j = \{x \in \Sigma_{\mathcal{B}} | x * \partial F_j = x\}$ with the subspace topology. For any $x \in X_j$, a filling f for x is a packing by K such that $f * \partial F_j = x$, $f * F_j = f$ and the number of elements of f is as large as possible given these constraints.

Lemma 2.2.4. *For each $j \geq 0$, there exists a Borel map $\phi_j : X_j \to \Sigma_{\mathcal{B}}$ such that for each $x \in X_j$, $\phi_j(x)$ is a filling for x.*

Proof. Let $j \geq 0$ be fixed and let $X = X_j$. Given $x \in X$, f a filling for x and $m > 0$, let $Y(f, m) = \{x' \in X|$ there exists a filling f' for x' with $D_K(f', f) < 2^{-m}\}$. From the compactness of F_j it follows that given $m > 0$, there exists a finite number of packings $x_1, ..., x_p \in X$ and fillings f_k for x_k such that $X = \bigcup_{k=1}^{p} Y(f_k, m)$.

Thus for a given integer $m > 0$, there exists a finite partition $\{A_{m,k}\}_{k=1}^{r_m}$ of X such that each $A_{m,k}$ is Borel and contained in some $Y(f, m)$. We will assume that $\{A_{m+1,k}\}_{k=1}^{r_{m+1}}$ refines $\{A_{m,k'}\}_{k'=1}^{r_m}$ (i.e. for every $A_{m+1,k}$, there exists an $A_{m,k'}$ such that $A_{m+1,k} \subset A_{m,k'}$). For each $m > 0$ and $1 \leq k \leq r_m$, choose $a(m, k) \in A_{m,k}$ and $f(m, k)$ a filling for $a(m, k)$ so that the following are satisfied:

1. $\{a(m,k)\}_{k=1}^{r_m} \subset \{a(m+1,k')\}_{k'=1}^{r_{m+1}}$ for all $m > 0$.

2. For every $m > 0$, $1 \leq k \leq r_m$, $a \in A_{m,k}$ there is a filling f for a such that $d_K(f, f(m,k)) < 2^{-m+1}$. This is possible since for each $A_{m,k}$ there exists $x \in X$ and a filling f for x with $A_{m,k} \subset Y(f,m)$.

3. If for some m and k, k', $A_{m+1,k'} \subset A_{m,k}$, then $d_K(f(m+1,k'), f(m,k)) < 2^{-m+1}$.

We define a map $\alpha_m : X \to \Sigma_{\mathcal{B}}$ by $\alpha_m(x) = f(m,k)$ if $x \in A_{m,k}$. Each α_m is Borel (since each $A_{m,k}$ is Borel). We claim that $\{\alpha_m\}_{m=1}^{\infty}$ converges pointwise as to a map ϕ. Let $x \in X$. Then by the third condition above, if $m < m'$ then

$$D_K(\alpha_m(x), \alpha_{m'}(x)) \leq \sum_{k=m}^{m'-1} 2^{-k+1} \leq \sum_{k=m}^{\infty} 2^{-k+1} \to 0 \qquad (2.32)$$

as $m \to \infty$. So the sequence $\{\alpha_m(x)\}_{m=1}^{\infty}$ is Cauchy and therefore converges in $\Sigma_{\mathcal{B}}$ to an element $\phi(x)$. Since all the maps α_m are Borel, ϕ must be Borel, too.

We claim that $\phi(x)$ is a filling for x for each $x \in X$. So let $x \in X$ and for each $m > 0$ choose a filling f_m for x such that $d_K(f_m, \alpha_m(x)) < 2^{-m+1}$. By the definition of $A_{m,k}$ and the second condition of $f(m,k)$ listed above, such an f_m exists for all $m > 0$. By the triangle inequality, $d_K(f_m, \phi(x)) \leq 2^{-m+1} + \sum_{k=m}^{\infty} 2^{-k+1}$ which goes to zero as m goes to infinity. Hence the sequence $\{f_m\}_{m>0}$ converges to $\phi(x)$. Since f_m is a filling for x for each m, it follows that $\phi(x)$ is also a filling for x. Now we are done.

$\square$

We choose maps Φ_j satisfying the conclusion of the above lemma. Define $\Phi_j : \Sigma_{\mathcal{B}} \to \Sigma_{\mathcal{B}}$ by the following properties:

1. for any $\mathcal{P} \in \Sigma_{\mathcal{B}}$, $\Phi_j(\mathcal{P}) * F_j = \phi_j(\mathcal{P} \cap \partial F_j) * F_j$;

2. for any $g \in G_j$ and $\mathcal{P} \in \Sigma_{\mathcal{B}}$, $\Phi_j(g\mathcal{P}) = g\Phi_j(\mathcal{P})$.

It should be clear from this definition that for each j, Φ_j is Borel. Let $\mu \in \mathcal{M}_I$ be given. Let $\mu'_j = \Phi_{j*}(\mu)$, i.e. for every Borel set $E \subset \Sigma_{\mathcal{B}}$, $\mu'_j(E) = \mu(\Phi_j^{-1}(E))$. By the above, μ'_j is a Borel probability measure that is invariant under $\mathcal{G}_j$. Let $\mu_j \in \mathcal{M}$ be defined by

$$\mu_j(E) = \frac{1}{vol(F_j)} \int_{\pi^{-1}(F_j)} \mu'_j(gE)\, d\lambda_{\mathcal{G}}(g) \tag{2.33}$$

for any Borel $E \subset \Sigma_{\mathcal{B}}$.

Lemma 2.2.5. *For every $h \in \mathcal{G}$ and Borel set $E \subset \Sigma_{\mathcal{B}}$, $\mu_j(hE) = \mu_j(E)$, i.e. $\mu_j \in \mathcal{M}_I$.*

Proof. Let $h \in \mathcal{G}$ and let $E \subset \Sigma_{\mathcal{B}}$ be a Borel set. We claim that for some k, there exists elements $g_1, ..., g_k \in \mathcal{G}_j$ and Borel subsets $F_j^1, ..., F_j^k \subset \pi^{-1}(F_j)$ such that

1. $\cup_{i=1}^k g_i F_j^i = \pi^{-1}(F_j)h$;

2. $\lambda_{\mathcal{G}}(F_j^{i_1} \cap F_j^{i_2}) = 0$ whenever $i_1 \neq i_2$;

3. $\lambda_{\mathcal{G}}(\cup_{i=1}^k F_j^i) = \lambda_{\mathcal{G}}(\pi^{-1}(F_j))$.

Since $\pi^{-1}(F_j)$ is a fundamental domain for the (left) action of $\mathcal{G}_j$ on $\mathcal{G}$ and $\pi^{-1}(F_j)h$ is compact, there are elements $g_1, ..., g_k \in \mathcal{G}_j$ such that $\pi^{-1}(F_j)h \subset \cup_{i=1}^k g_i\pi^{-1}(F_j)$. We let $F_j^i = g_i^{-1}(g_i\pi^{-1}(F_j) \cap \pi^{-1}(F_j)h)$. By definition then, the first part of the claim is true.

Suppose $g \in F_j^{i_1} \cap F_j^{i_2}$ for some $i_1, i_2 \in \{1, ..., k\}$ with $i_1 \neq i_2$. Then $g_{i_1}g$ and $g_{i_2}g$ are in $\pi^{-1}(F_j)h \cap \mathcal{G}_j g$. So $g_{i_1}gh^{-1}$ and $g_{i_2}gh^{-1}$ are in $\pi^{-1}(F_j) \cap \mathcal{G}_j gh^{-1}$. Since $\mathcal{G}_j g_{i_1}gh^{-1} = G_j g_{i_2}gh^{-1}$ and $\pi^{-1}(F_j)$ is a fundamental domain for $\mathcal{G}_j$, we must have that $g_{i_1}gh^{-1}$ and $g_{i_2}gh^{-1}$ are on the boundary of $\pi^{-1}(F_j)$. Hence $g \in g_{i_1}^{-1}\partial[\pi^{-1}(F_j)]h$. Since g is arbitrary, $F_j^{i_1} \cap F_j^{i_2} \subset g_{i_1}^{-1}\partial[\pi^{-1}(F_j)]h$. Since $\lambda_G(\partial[\pi^{-1}(F_j)]) = 0$ the second part of the claim follows. The third part of the claim follows directly from the first two parts and the fact that $\lambda_{\mathcal{G}}$ is right-invariant as well as left-invariant (i.e. $\lambda_{\mathcal{G}}(Eg) = \lambda_{\mathcal{G}}(E)$ for any Borel $E \subset \mathcal{G}$ and $g \in \mathcal{G}$).

To finish the proof, note

$$\begin{aligned}
\mu_j(hE) &= \frac{1}{vol(F_j)} \int_{\pi^{-1}(F_j)} \mu_j'(ghE)\, d\lambda_{\mathcal{G}}(g) && (2.34)\\
&= \frac{1}{vol(F_j)} \int_{\cup_i g_i F_j^i h^{-1}} \mu_j'(ghE)\, d\lambda_{\mathcal{G}}(g) && (2.35)\\
&= \frac{1}{vol(F_j)} \sum_{i=1}^{k} \int_{g_i F_j^i h^{-1}} \mu_j'(ghE)\, d\lambda_{\mathcal{G}}(g) && (2.36)\\
&= \frac{1}{vol(F_j)} \sum_{i=1}^{k} \int_{F_j^i} \mu_j'(g_i gE)\, d\lambda_{\mathcal{G}}(g) && (2.37)\\
&= \frac{1}{vol(F_j)} \sum_{i=1}^{k} \int_{F_j^i} \mu_j'(gE)\, d\lambda_{\mathcal{G}}(g) && (2.38)\\
&= \frac{1}{vol(F_j)} \int_{\pi^{-1}(F_j)} \mu_j'(gE)\, d\lambda_{\mathcal{G}}(g) && (2.39)\\
&= \mu_j(E). && (2.40)
\end{aligned}$$

The fourth equation above holds by the change of variables theorem and because $\lambda_{\mathcal{G}}$ is right-invariant. The fifth equation holds because μ'_j is invariant under $\mathcal{G}_j$. $\square$

The next two lemmas will provide tools for calculating $D(\mu)$ and $D(\mu_j)$ with respect to the relative density within F_j.

Lemma 2.2.6.

$$D(\mu_j) = \int_{\Sigma_{\mathcal{B}}} \frac{vol(\mathcal{P} \cap F_j)}{vol(F_j)}\, d\mu'_j(\mathcal{P}) \tag{2.41}$$

Proof. Let $J = \{\mathcal{P} \in \Sigma_{\mathcal{B}} | \mathcal{O} \in \mathcal{P}\}$. Let χ_J denote the characteristic function of J. By definition of density and of μ_j,

$$\begin{aligned} D(\mu_j) &= \mu_j(J) && (2.42)\\ &= \frac{1}{vol(F_j)} \int_{\pi^{-1}(F_j)} \mu'_j(gJ)\, d\lambda_{\mathcal{G}}(g) && (2.43)\\ &= \frac{1}{vol(F_j)} \int_{\pi^{-1}(F_j)} \int_{\Sigma_{\mathcal{B}}} \chi_{gJ}(\mathcal{P})\, d\mu'_j(\mathcal{P})\, d\lambda_{\mathcal{G}}(g) && (2.44)\\ &= \int_{\Sigma_{\mathcal{B}}} \frac{1}{vol(F_j)} \int_{\pi^{-1}(F_j)} \chi_{gJ}(\mathcal{P})\, d\lambda_{\mathcal{G}}(g)\, d\mu'_j(\mathcal{P}) && (2.45) \end{aligned}$$

Hence we will be done once we show that for any $\mathcal{P} \in \Sigma_{\mathcal{B}}$,

$$\int_{\pi^{-1}(F_j)} \chi_{gJ}(\mathcal{P})\, d\lambda_{\mathcal{G}}(g) = vol(\mathcal{P} \cap F_j). \tag{2.46}$$

So,

$$
\begin{aligned}
\int_{\pi^{-1}(F_j)} \chi_{gJ}(\mathcal{P})\, d\lambda_{\mathcal{G}}(g) &= \lambda_{\mathcal{G}}(\{g \in \pi^{-1}(F_j) |\, \mathcal{P} \in gJ\}) && (2.47)\\
&= \lambda_{\mathcal{G}}(\{g \in \pi^{-1}(F_j) |\, g^{-1}\mathcal{P} \in J\}) && (2.48)\\
&= \lambda_{\mathcal{G}}(\{g \in \pi^{-1}(F_j) |\, \mathcal{O} \in c(g^{-1}\mathcal{P})\}) && (2.49)\\
&= \lambda_{\mathcal{G}}(\{g \in \pi^{-1}(F_j) |\, g\mathcal{O} \in \mathcal{P}\}) && (2.50)\\
&= \lambda_{\mathcal{G}}(\pi^{-1}(\mathcal{P} \cap F_j)) && (2.51)\\
&= vol(\mathcal{P} \cap F_j) && (2.52)
\end{aligned}
$$

The last equation holds since for any Borel set $E \subset \mathbb{S}$, $vol(E) = \lambda_{\mathcal{G}}(\pi^{-1}(E))$. □

Lemma 2.2.7.

$$
D(\mu) = \int_{\Sigma_{\mathcal{B}}} \frac{vol(\mathcal{P} \cap F_j)}{vol(F_j)}\, d\mu \tag{2.53}
$$

Proof. By equation (20) in the previous lemma,

$$
\begin{aligned}
\int_{\Sigma_{\mathcal{B}}} \frac{vol(\mathcal{P} \cap F_j)}{vol(F_j)}\, d\mu(\mathcal{P}) &= \int_{\Sigma_{\mathcal{B}}} \int_{\pi^{-1}(F_j)} \frac{\chi_J(g\mathcal{P})}{vol(F_j)}\, d\lambda_{\mathcal{G}}(g)\, d\mu(\mathcal{P}) && (2.54)\\
&= \int_{\pi^{-1}(F_j)} \int_{\Sigma_{\mathcal{B}}} \frac{\chi_J(g\mathcal{P})}{vol(F_j)}\, d\mu(\mathcal{P})\, d\lambda_G(g) && (2.55)\\
&= \int_{\pi^{-1}(F_j)} \int_{\Sigma_{\mathcal{B}}} \frac{\chi_J(\mathcal{P})}{vol(F_j)}\, d\mu(\mathcal{P})\, d\lambda_{\mathcal{G}}(g) && (2.56)\\
&= \int_{\pi^{-1}(F_j)} \frac{D(\mu)}{vol(F_j)} d\lambda_{\mathcal{G}}(g) && (2.57)\\
&= D(\mu) && (2.58)
\end{aligned}
$$

The third inequality above holds because μ is $\mathcal{G}$-invariant. □

Proof. (of Theorem 2.2.3) Let U_j be the set of all packings in $\Sigma_{\mathcal{B}}$ that are unsaturated relative to F_j, i.e. for any $\mathcal{P} \in U_j$, there is a packing $\mathcal{P}'$ such that $\mathcal{P} * \partial F_j = \mathcal{P}' * \partial F_j$ but $\mathcal{P}' * F_j$ has greater volume than $\mathcal{P} * F_j$. Because $\mathcal{B}$ is a finite collection, for each j, there exists a constant $c_j > 0$ such that if $\mathcal{P} \in U_j$ and f is a filling for $\mathcal{P}$, then $vol(f \cap F_j) \geq vol(\mathcal{P} \cap F_j) + c_j$.

$$
\begin{aligned}
D(\mu_j) &= \int_{\Sigma_{\mathcal{B}}} \frac{vol(\mathcal{P} \cap F_j)}{vol(F_j)}\, d\mu_j'(\mathcal{P}) && (2.59)\\
&= \int_{\Sigma_{\mathcal{B}}} \frac{vol(\Phi_j(\mathcal{P}) \cap F_j)}{vol(F_j)}\, d\mu(\mathcal{P}) && (2.60)\\
&= \int_{U_j} \frac{vol(\Phi_j(\mathcal{P}) \cap F_j)}{vol(F_j)}\, d\mu(\mathcal{P}) && (2.61)\\
&+ \int_{\Sigma_{\mathcal{B}}-U_j} \frac{vol(\Phi_j(\mathcal{P}) \cap F_j)}{vol(F_j)}\, d\mu(\mathcal{P}) && (2.62)\\
&\geq \int_{U_j} \frac{vol(\mathcal{P} \cap F_j) + c_j}{vol(F_j)}\, d\mu(\mathcal{P}) && (2.63)\\
&+ \int_{\Sigma_{\mathcal{B}}-U_j} \frac{vol(\Phi_j(\mathcal{P}) \cap F_j)}{vol(F_j)}\, d\mu(\mathcal{P}) && (2.64)\\
&= \int_{\Sigma_{\mathcal{B}}} \frac{vol(\mathcal{P} \cap F_j)}{vol(F_j)}\, d\mu(\mathcal{P}) + \mu(U_j)\frac{c_j}{vol(F_j)} && (2.65)\\
&= D(\mu) + \mu(U_j)\frac{c_j}{vol(F_j)} && (2.66)
\end{aligned}
$$

The first equality is Lemma 2.2.6, the second comes from the definition of μ_j' and the last equality uses lemma 2.2.7.

If μ is optimally dense then by definition $D(\mu) \geq D(\mu_j)$. So, $\mu(U_j) = 0$ for all j. Hence $\mu(\bigcup_j U_j) = 0$. Since $B_j \subset F_j$ for all j, $U = \bigcup_j U_j$ is the set of all packings that are not completely saturated. $\square$

2.2.2 Locally densest does not imply globally densest

As pointed out in [12], a completely saturated packing of Euclidean space is a densest packing. This is not true in hyperbolic space. In this example, we construct a pair of bodies β_1, β_2 in $\mathbb{H}^2$ such that two completely saturated periodic packings by $\{\beta_1, \beta_2\}$ exist that have different densities.

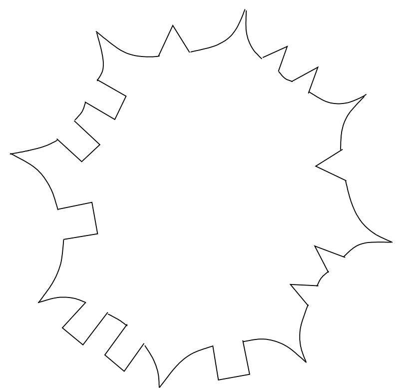

Figure 2.1: β_2'

The reason, as you will see, is due to the fact that the length of the boundary of a region in the hyperbolic plane is comparable to its area. Let be a regular octagon with all interior angles equal to $2\pi/8$. Let $\mathcal{T}_1$ be the unique periodic tiling by β_1. Let β_2' be the tile shown in figure 2.1. It is formed from β_1 by adding "protrusions" to some edges and "identations" to others. We will assume that these protrusions and indentations are made so that they fit together but are narrow enough so that there is a region of finite area C_1 in each indentation that cannot be occupied by a nonoverlapping copy of β_2' unless it is occupied by a protrusion. Also we assume that each protrusion fits into a unique indentation.

β_2' admits a unique periodic tiling $\mathcal{T}_2$. Let β_2 be equal to β_2' with a small hole removed from its interior. Let $\mathcal{P}$ be the obvious periodic packing by β_2 (i.e. the one that comes from $\mathcal{T}_2$ by removing a small hole from the interior of each tile). Since β_1 admits a periodic tiling, it is clear that the optimal density of $\{\beta_1, \beta_2\}$ is one. Just as clear, is the fact that the density of $\mathcal{P}$ is $area(\beta_2)/area(\beta_1) < 1$. We will show that $\mathcal{P}$ is completely saturated (if the hole in β_2 is small enough).

It is a standard fact of hyperbolic geometry that there exists a constant $C_2 > 0$ depending only on the symmetry group of $\mathcal{T}_1$ (and the fact that β_1 contains a fundamental domain for this group) such that for all finite subtilings $\mathcal{T}'$ of β_1, $|\partial\mathcal{T}'| \geq C_2|\mathcal{T}'|$ (by $|\partial\mathcal{T}'|$ we mean the number of edges contained in exactly one tile of $\mathcal{T}'$ and by $|\mathcal{T}'|$ we mean the number of tiles in $\mathcal{T}'$). Since the hole in the interior of β_2' can be made as small as we like, we may assume that $area(\beta_2) > area(\beta_1) - C_1C_2/2$.

Suppose for a contradiction that $\mathcal{P}$ is not completely saturated. Then there exists a finite subpacking $\mathcal{P}' \subset \mathcal{P}$ and another finite packing $\mathcal{P}''$ such that $(\mathcal{P} - \mathcal{P}') \cup \mathcal{P}''$ is a packing and $area(\mathcal{P}'') > area(\mathcal{P}')$. We may assume without loss of generality that $\mathcal{P} \cap \mathcal{P}'' = \emptyset$.

We claim that the number of edges of $\mathcal{P}'$ that have protrusions on them coming from bodies of $\mathcal{P}'$ is at least $|\partial\mathcal{P}'|/2$. So let e be any edge on the boundary of $\mathcal{P}'$. Let $e = e_0, e_1, .., e_n$ be the sequence of edges defined by for $1 \leq i < n$, e_{i+1} and e_i are on a body of $\mathcal{P}'$ and e_{i+1} is the "opposite side" of e_i in the sense that if e_i has a protrusion on it (relative to the body containing both e_i and e_{i+1}) then e_{i+1} is its corresponding indentation and vice versa.

This sequence is uniquely defined and ends in an edge e_n on the boundary of $\mathcal{P}'$. It is easy to see that if e_0 corresponds to an indentation of $\mathcal{P}'$ (i.e. e_0 has an indentation on it coming from a body of $\mathcal{P}'$) then e_n corresponds to a protrusion and vice versa. Thus the claim is proven.

Note that it is not possible for any body of $\mathcal{P}''$ to fill completely any indentation on the boundary of $\mathcal{P} - \mathcal{P}'$ (in fact a region of area at least C_1 is always unfilled). Hence the total area of $\mathcal{P}''$ is at most

$$
\begin{aligned}
area(\mathcal{P}'') &\leq |\mathcal{P}'|area(\beta_1) - (C_1/2)|\partial\mathcal{P}'| && (2.67)\\
&\leq |\mathcal{P}'|[area(\beta_1) - C_1C_2/2] && (2.68)\\
&< |\mathcal{P}'|area(\beta_2) && (2.69)\\
&= area(\mathcal{P}'). && (2.70)
\end{aligned}
$$

This contradicts the choice of $\mathcal{P}''$. So $\mathcal{P}$ is completely saturated. The moral is that, in hyperbolic space, locally densest does not imply globally densest.

2.3 What Invariant Measures avoid

Some packings, such as the Böröczky example, do not have a well-defined density. We claim this is due to the fact that the closure of the orbit of such a packing has measure zero with respect to every invariant measure μ. In this section we prove this statement and show other examples of packings that are not "seen" by invariant measures.

2.3.1 Penrose's binary tilings

Perhaps the most relevant to the discussion in section I is the $\mathcal{B}$ consisting of the body β shown in Figure 3. (This is a minor variation on the tile in [21], and a special case of tiles in [18].) We know copies of this body can tile $\mathbb{H}^2$, and since limits in $\Sigma_{\mathcal{B}}$ of tilings will again be tilings, if there were any invariant measure $\mu \in \mathcal{M}_I(K)$ with support in the orbit closure of such a tiling it would clearly have density 1. However we can see there is no such measure as follows. First consider the slightly simpler, and better known, example of the natural action of the isometry group $\mathcal{G}$ of $\mathbb{H}^2$ (namely $\mathcal{G} = PSL_2(\mathcal{R})$) on the boundary Δ of $\mathbb{H}^2$. Assume there is a measure μ on Δ invariant under $\mathcal{G}$. Any hyperbolic element $g_h \in \mathcal{G}$ has 2 fixed points in Δ, p_1, p_2, and moves all other points towards one and away from the other. From its invariance under g_h, $\mu(\{p_1, p_2\}) = 1$. Then considering that any elliptic element $g_e \in \mathcal{G}$ has no fixed points in Δ, and μ must also be invariant under g_e, we get a contradiction. So there are no probability measures on Δ invariant under $\mathcal{G}$. Going back to our space $\Theta_{\mathcal{B}}$ of tilings by our body β, consider the function f from $\Theta_{\mathcal{B}}$ to Δ, which takes each tiling to the point "pointed to" by the protrusion on each body in the tiling. f is obviously continuous. If there were a probability measure μ on the space $\Theta_{\mathcal{B}}$ of tilings, invariant under the action of $\mathcal{G}$, we could define a corresponding measure μ_f on Δ by $\mu_f(E) = \mu(f^{-1}[E])$. Since no such μ_f exists, this proves no such μ exists.

Now assume the optimal density for $\mathcal{B}$, $D(\mathcal{B})$, is 1, with an optimal measure μ. For each $R > 0$ consider the function on $\Sigma_{\mathcal{B}}$

$$f_R(\mathcal{P}) = \frac{1}{vol(B_R)} \int_{\tilde{B}_R} F_{\mathcal{O}}(g\mathcal{P})\, d\lambda_{\mathcal{G}}(g), \tag{2.71}$$

which gives the relative area of the ball B_R covered by the disks of $\mathcal{P}$. From the invariance of of μ,

$$\int_{\Sigma_{\mathcal{B}}} f_R(\mathcal{P})\, d\mu(\mathcal{P}) = \int_{\Sigma_{\mathcal{B}}} F_{\mathcal{O}}(\mathcal{P})\, d\mu(\mathcal{P}) = D(\mu) = 1, \tag{2.72}$$

so $f_R(\mathcal{P}) = 1$ for μ-almost every $\mathcal{P} \in \Sigma_{\mathcal{B}}$. Letting R run through the positive integers, and intersecting the sets of full measure we get for each such R, we see there is a set of packings of full measure which are tilings. Since the closure of a set of tilings can only contain tilings, and the support of μ must be invariant under $\mathcal{G}$, that support is contained in the set of all tilings of β. But we saw above that there can be no such measure as μ, and this proves that $D(\mathcal{B}) \neq 1$.

Using modifications of the above example, it can be shown [8] that for every $\epsilon > 0$, there exists a body β that admits a tiling of $\mathbb{H}^n$ but $D(\beta) < \epsilon$.

The formalism above leads one to assert that the densest packings of the body β have density bounded away from 1, even though one can tile $\mathbb{H}^2$ with copies of β. The "reason" for this is that there are no invariant measures which can "see" the tilings; they are a set of measure zero for every invariant measure on the space of all packings by β. We explore the consequences of this using some of the examples we discussed earlier.

Consider the tile β shown in Figure 2.2. Congruent copies of β can tile the plane, in only one way up to an overall rigid motion, as in Figure 2.3 (in which the little bumps on the tiles are not shown.)

Construct the tile $\bar{\beta}$ of Figure 2.4 out of three abutting copies of β. Now drill a hole in $\bar{\beta}$, producing the body $\bar{\beta}_0$, as shown in Figure 2.5. Note that the packings of the plane by $\bar{\beta}_0$ obtained in the obvious way from the

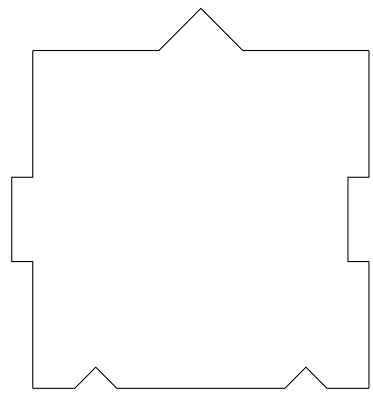

Figure 2.2: A "binary" tile

tilings by $\bar{\beta}$, are precisely the complements of the disk packings of Böröczky discussed above. The point is, although it might seem reasonable to assign a density of 1 to the tiling of Figure 2.3, that would seem to imply a well defined density to the packing of Figure 1.2, which we know is misleading. In other words, the meaningfulness of the density of the tiling of Figure 2.3 is unstable under arbitrarily small perturbations (drilling arbitrarily small holes). Notice that when we drill these small holes we turn the tiling into a mere packing, forcing us to give up the "simplicity" of the tiling, as a global object with seemingly obvious density, and leaving us to find some meaningful way to assign a density to the resulting packing. As we will see below, the difficulty in assigning a density to a packing, for instance congruent copies of a single body β, can derive from the complexity of the set of rigid motions of β that define the packing. And in this sense a tiling is no simpler; treating it as a global object with an "obvious" density simply avoids coming to grips with the essential nature of the assignment of density for packings.

In other words, the phenomenon whereby the "optimal" density can be

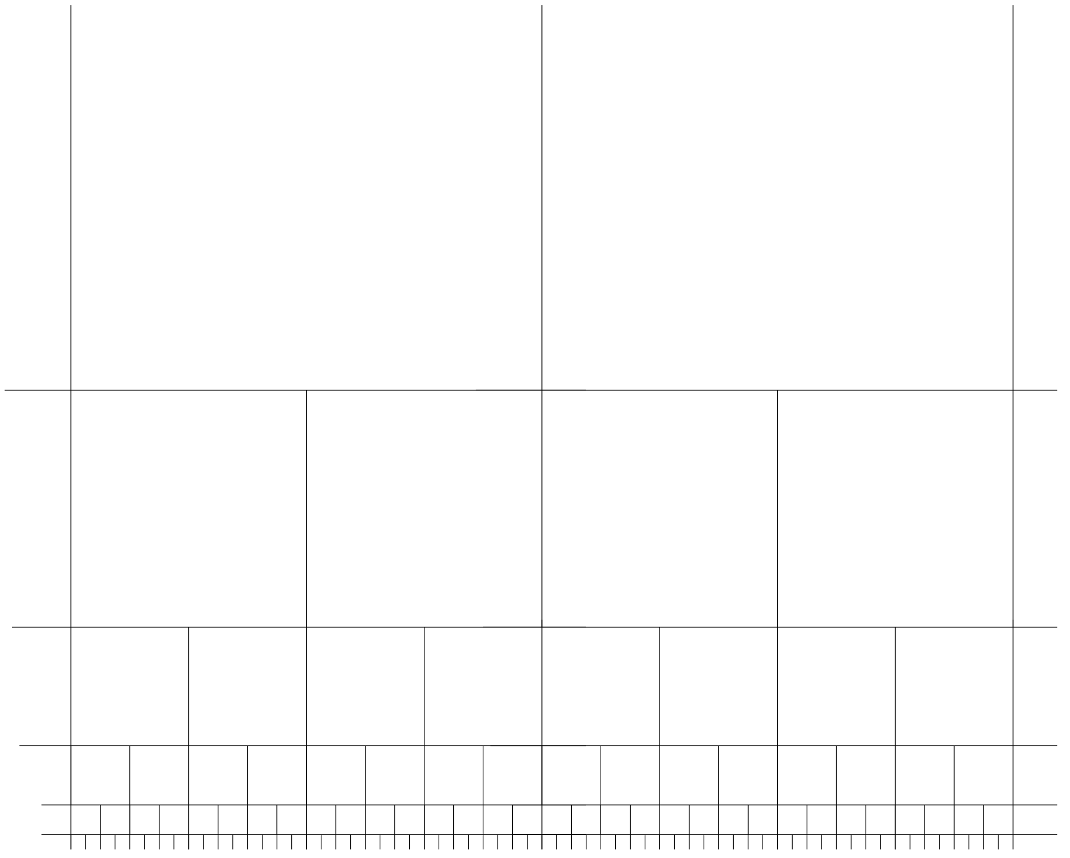

Figure 2.3: A “binary” tiling

less (even far less) than 1 for a body which can tile space, can be understood as related to the instability of the meaningfulness of the density of the tilings under removal of small holes in the tiles. This suggest that even for tilings one needs to keep track of the individuality of the tiles. In this example that amounts to noting the various sets of congruences used in producing the tilings; in some sense those sets of congruences are too complicated to be analyzed through the formalism.

We have shown that if $\mathcal{T}$ is a tiling whose orbit closure in the space of

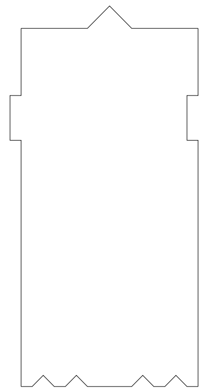

Figure 2.4: A tile

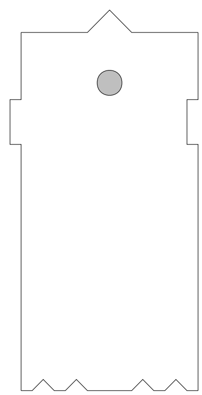

Figure 2.5: A body

packings factors onto the space at infinity of the hyperbolic plane, then there are no invariant measures on the orbit closure of $\mathcal{T}$. All of our examples of strange behaviour in the hyperbolic plane have, so far, been constructed using this principle. Could this be the only way of constructing such examples?

2.3.2 A different tiling without an invariant measure

Let τ be the tile shown in figure 2.6. It is an all right octagon whose sides come in two different lengths and there is a protrusion on one of the short sides and

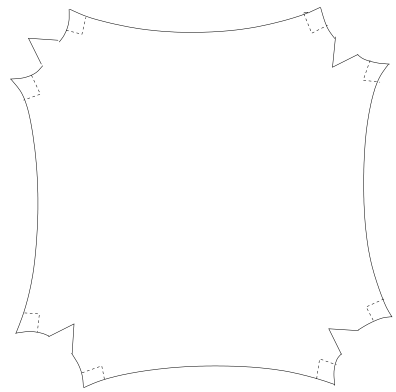

Figure 2.6: τ

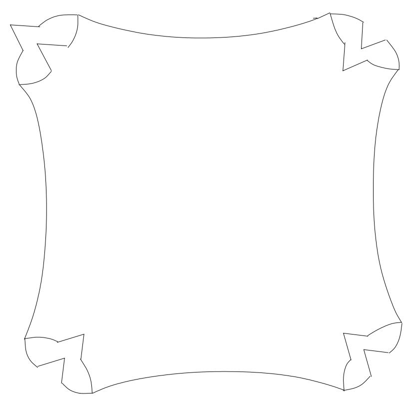

Figure 2.7: X

an indentation on each of the other three short sides. Two copies of τ can be joined along their long sides to produce an X-tile (see figure 2.7). Many copies of this X tile can be glued together to form a noncompact surface S without boundary. There is in fact, only one way to do this. S is homeomorphic to the boundary of a regular neighborhood of the standard Cayley graph for the free group on two generators embedded in $\mathcal{R}^3$. The hyperbolic plane isometrically covers S and therefore the tiling $\mathcal{T}_S$ of S by copies of τ lifts to a tiling $\mathcal{T}$ of $\mathbb{H}^2$ by copies of τ.

Note that the free group F_2 on 2 generators $\{a, b\}$ acts naturally and isometrically on S. This action lifts to an isometric action of $\mathbb{H}^2$ via the covering map. Though not relevant to what follows, note that this lift is unique up to postcomposition by rigid motions of $\mathbb{H}^2$.

If we start a walk in $\mathcal{T}_S$ from some initial tile and follow the protrusions we get "closer" to a point on the ideal boundary of S. It is not too hard to see that this point does not depend on the initial tile chosen but only on the tiling $\mathcal{T}_S$. Therefore, there is a map from the space of tilings of S by X (defined similar to the same way $\Sigma_{\mathcal{B}}$ is defined) to the ideal boundary of S that commutes with the action of F_2. Since the ideal boundary does not admit an invariant Borel probability measure (for practically the same reason that Δ does not admit an invariant measure), neither does the space of tilings of S by X.

Suppose that there exists an invariant measure μ whose support is contained in the orbit closure of $\mathcal{T}$. Then this measure pushes forward via the covering map to a measure μ_S on the space of tilings on S by τ. This measure μ_S is invariant under the action of F_2 but this contradicts the previous paragraph.

Now suppose that there is an equivariant map ϕ from the orbit closure $\bar{O(\mathcal{T})}$ of $\mathcal{T}$ in Σ_τ to Δ. Let $p = \phi(\mathcal{T})$. Since ϕ is equivariant, the stabilizer of $\mathcal{T}$ must be contained in the stabilizer of p. However, the stabilizer of $\mathcal{T}$ is noncyclic (since it contains an isomorphic copy of the fundamental group of S which is noncyclic). By the theory of fuchsian groups, $\mathcal{T}$ does not fix any point at infinity. This contradiction shows that ϕ cannot exist.

Chapter 3

Sphere Packings

In this chapter, we prove that for most radii r, all optimally dense measures for the ball of radius r are nonperiodic. By abuse of notation, we will write $\Sigma_r = \Sigma_{B_r}$ where B_r is the ball of radius r.

Theorem 3.0.1. *The set of all r such that there exists a periodic optimally dense measure μ on Σ_r is at most countable.*

The proof will follow quickly after the preparation of the next four lemmas.

Lemma 3.0.2. *Suppose $\mathcal{P}$ is a periodic packing. If $\mu_{\mathcal{P}}$ is optimally dense, then $\mathcal{P}$ is completely saturated.*

Proof. By theorem 2.2.3, there exists a set Z of full $\mu_{\mathcal{P}}$ measure such that each packing in Z is completely saturated. Since Z is of full measure, there exists a packing $\mathcal{P}$ in Z that is in the support of $\mu_{\mathcal{P}}$. But the support of $\mu_{\mathcal{P}}$ is equal to the orbit of $\mathcal{P}$. Hence $\mathcal{P}$ is completely saturated.

□

For $a, q \in \mathbb{H}^n$ and $s > 0$, we define $a_q(s)$ to be the point on the ray from q to a whose distance from q is equal to $sd(q, a)$.

Lemma 3.0.3. *Let $q \in \mathbb{H}^n$. If a, b are any distinct points of $\mathbb{H}^n$ but neither equal to q and $s > 0$ then $d(a,b) \leq d(a_q(s), b_q(s))$.*

Proof. Let $\angle aqb$ denote the acute angle between aq and qb. Define a function H by

$$H(y,z,s) = \cosh(y+s)\cosh(z+s) - \cos(\angle aqb)\sinh(y+s)\sinh(z+s). \quad (3.1)$$

Then, by the law of cosines (see [24]), $H(d(a,q), d(b,q), s) = \cosh(d(a_q(s), b_q(s)))$. So it suffices to show that the derivative of H at (y,z,s) with respect to s is positive whenever all the variables y, z and s are positive. So we compute:

$$\begin{aligned}
dH/ds &= \sinh(y+s)\cosh(z+s) + \cosh(y+s)\sinh(z+s) && (3.2)\\
&- \cos(\angle aqb)[\cosh(y+s)\sinh(z+s) && (3.3)\\
&\quad + \sinh(y+s)\cosh(z+s)] && (3.4)\\
&= (1-\cos(\angle aqb))[\cosh(y+s)\sinh(z+s) && (3.5)\\
&\quad + \sinh(y+s)\cosh(z+s)] && (3.6)\\
&\geq 0. && (3.7)
\end{aligned}$$

□

We will denote the set of centers of a sphere packing $\mathcal{P}$ by $C_{\mathcal{P}}$ and the radius of the spheres in $\mathcal{P}$ by $rad(P)$.

Lemma 3.0.4. *Suppose $\mathcal{P}$ is a packing for which there exists $t > 0$ such that the distance between any two centers of $\mathcal{P}$ is greater than or equal to $2rad(\mathcal{P}) + t$. Then $\mathcal{P}$ is not completely saturated.*

Proof. The idea behind the proof is that by moving a finite number of balls away from a point $q \notin C_{\mathcal{P}}$, there will be space enough to place a new ball with center at q.

Let k be the integer such that $kt > 2rad(\mathcal{P}) \geq (k-1)t$ and let $R = k(2rad(\mathcal{P}) + t)$. We define a function f on $C_{\mathcal{P}}$ as follows.

(i) If $d(c, q) \geq R$ then let $f(c) = c = c_q(0)$.

(ii) Otherwise there is a j such that $0 < j \leq k$ and $R - 2(j-1)(rad(\mathcal{P}) + t) > d(q, c) \geq R - 2j(rad(\mathcal{P}) + t)$. In this case, define $f(c) = c_q(jt)$.

Let C' be the union of the point q and the image of $C_{\mathcal{P}}$ under f. From the definitions, it is clear that C' differs from $C_{\mathcal{P}}$ in only a finite number of points. Hence once we show that balls of radius $rad(\mathcal{P})$ centered at points of C' do not overlap, it follows that $\mathcal{P}$ is not completely saturated.

Note that if $c \in C_{\mathcal{P}}$ and $f(c) = c_q(jt)$ then

$$\begin{aligned} d(f(c), q) &= d(c_q(jt), q) && (3.8) \\ &= d(c, q) + jt && (3.9) \\ &\geq R - 2j[rad(\mathcal{P}) + t] + jt && (3.10) \\ &= (k-j)[2rad(\mathcal{P}) + t]. && (3.11) \\ & && (3.12) \end{aligned}$$

Hence if the ball of radius $rad(\mathcal{P})$ centered at $f(c)$ overlaps the ball of radius $rad(\mathcal{P})$ centered at q then $f(c) = c_q(kt)$. But this implies that $d(q, f(c)) > kt > 2rad(\mathcal{P})$, a contradiction. Hence the balls of radius $rad(\mathcal{P})$ centered at q and $f(c)$ do not overlap for any $c \in C_{\mathcal{P}}$.

Now assume for a contradiction that there exist two distinct centers c and c' in $C_{\mathcal{P}}$ with $d(f(c), f(c')) < 2rad(\mathcal{P})$. If $d(c,q) \geq R$ and $d(c',q) \geq R$ then $f(c) = c$ and $f(c') = c'$ so $d(f(c), f(c')) = d(c,c') > 2rad(\mathcal{P})$, a contradiction. So we may assume that $d(q,c') < R$. Suppose that $f(c) = c_q(jt)$ and $f(c') = c'_q(j't)$. Note by the distance inequality, the assumption that $d(f(c), f(c')) < 2rad(\mathcal{P})$ and the definition of f,

$$\begin{aligned} d(f(c),q) &\leq d(f(c), f(c')) + d(f(c'),q) && (3.13)\\ &< 2rad(\mathcal{P}) + d(c',q) + j't && (3.14)\\ &< 2rad(\mathcal{P}) + R - 2(j'-1)[rad(\mathcal{P}) + t] + j't. && (3.15) \end{aligned}$$

Next by definition of R,

$$2rad(\mathcal{P}) + R - 2(j'-1)[rad(\mathcal{P}) + t] + j't = (k - j' + 2)[2rad(\mathcal{P}) + t]. \quad (3.16)$$

The previous equations (3.8), (3.13) and (3.16) now imply that

$$0 < (j - j' + 2)[2rad(\mathcal{P}) + t]. \quad (3.17)$$

So

$$j' < j + 2. \quad (3.18)$$

If $j = 0$ then $j' = 1$. Otherwise $d(q,c) < R$ and so by symmetry,

$$j < j' + 2. \quad (3.19)$$

In any case, we may assume that either $j' = j$ or $j' = j + 1$. So Lemma 3.0.3 shows that

$$
\begin{aligned}
2rad(\mathcal{P}) + t &\leq d(c, c') && (3.20)\\
&\leq d(c_q(jt), c'_q(jt)) && (3.21)\\
&= d(f(c), c'_q(jt)) && (3.22)\\
&\leq d(f(c), c'_q(j't)) + d(c'_q(j't), c'_q(jt)) && (3.23)\\
&= d(f(c), f(c')) + d(c'_q(j't), c'_q(jt)) && (3.24)\\
&< 2rad(\mathcal{P}) + t. && (3.25)
\end{aligned}
$$

This contradiction finishes the lemma. □

For a packing $\mathcal{P}$, let $K(\mathcal{P})$ be the m-complex underlying the Delone cell decomposition of $\mathcal{P}$. In other words, the vertex set of $K(\mathcal{P})$ is equal to the set of ball centers of $\mathcal{P}$, an edge exists between vertices v_1 and v_2 if and only an edge of a Delone cell connects the corresponding centers in $\mathcal{P}$, and so on. Let $Aut(\mathcal{P})$ denote the automorphism group of $K(\mathcal{P})$, i.e. the group of bijective maps from $K(\mathcal{P})$ to $K(\mathcal{P})$ that preserve its structure as an m-complex.

Lemma 3.0.5. *If $\mathcal{P}_0$ and $\mathcal{P}_1$ are periodic packings, $K(\mathcal{P}_0)$ and $K(\mathcal{P}_1)$ are isomorphic as m-complexes, and $rad(\mathcal{P}_0) < rad(\mathcal{P}_1)$, then $\mu_{\mathcal{P}_0}$ is not optimally dense.*

Proof. Let $\mathcal{P}_2$ be the packing such that $C_{\mathcal{P}_2} = C_{\mathcal{P}_1}$ and $rad(\mathcal{P}_2) = rad(\mathcal{P}_0)$. Intuitively, $\mathcal{P}_2$ is formed from $\mathcal{P}_1$ by shrinking the radius of the balls to $rad(\mathcal{P}_0)$. Let $t = 2rad(\mathcal{P}_1) - 2rad(\mathcal{P}_0) > 0$. By Lemma 3.0.4 $\mathcal{P}_2$ is not completely saturated. By Lemma 3.0.2 $\mu_{\mathcal{P}_2}$ is not optimally dense. We will show that

$D(\mu_{\mathcal{P}_2}) = D(\mu_{\mathcal{P}_0})$. Given this it follows that $\mu_{\mathcal{P}_0}$ is not optimally dense, which proves the lemma.

Note that $K(\mathcal{P}_1) = K(\mathcal{P}_2)$, so $K(\mathcal{P}_0)$ and $K(\mathcal{P}_2)$ are isomorphic as m-complexes. Hence there exists a homeomorphism $\Phi : \mathbb{H}^n \to \mathbb{H}^n$ such that Φ take the k-cells of $K(\mathcal{P}_0)$ to the k-cells of $K(\mathcal{P}_2)$ for $0 \leq k \leq m$ and vice-versa. Note that Φ induces an isomorphism Φ_* from $Aut(\mathcal{P}_0)$ to $Aut(\mathcal{P}_2)$ by $\Phi_*(\alpha)(u) = \Phi\alpha\Phi^{-1}(u)$ for any $\alpha \in Aut(\mathcal{P}_2)$ and $u \in K(\mathcal{P}_2)$. Also there are natural (injective) inclusion homorphisms $i_j : G_{\mathcal{P}_j} \to Aut(\mathcal{P}_j)$ for $j = 0, 2$. Since $G_{\mathcal{P}_j}$ is cocompact (for $j = 0, 2$), $i_j(G_{\mathcal{P}_j})$ has finite index in $Aut(\mathcal{P}_j)$. Let G_t be a finite-index torsion-free subgroup of $G_{\mathcal{P}_0}$. The group $\hat{H}_2 \equiv \Phi_* i_0(G_t) \cap i_2(G_{\mathcal{P}_2})$ has finite index in $Aut(\mathcal{P}_2)$. So $\hat{H}_0 \equiv \Phi_*^{-1}(H_2)$ has finite index in $Aut(\mathcal{P}_0)$. So $H_j = i_j^{-1}(\hat{H}_j)$ for $j = 0, 2$ has finite index in $G_{\mathcal{P}_j}$. Note both H_0 and H_2 are torsion free. By the Gauss-Bonnet theorem in dimension 2 and Mostow rigidity (see [24]) in higher dimensions, $H_0\backslash\mathbb{H}^n$ has the same volume as $H_2\backslash\mathbb{H}^n$. By the homeomorphism Φ, $H_0\backslash K(\mathcal{P}_0)$ has the same number of vertices as $H_1\backslash K(\mathcal{P}_1)$. Hence $H_0\backslash\mathcal{P}_0$ has the same number of balls as $H_2\backslash\mathcal{P}_2$. Proposition 2.0.2 now implies that $D(\mu_{\mathcal{P}_0}) = D(\mu_{\mathcal{P}_2})$.

□

Proof. (of theorem 3.0.1) Lemma 3.0.5 implies that the set of all radii that admit an optimally dense periodic measure injects into the set of finite m-complexes (by a map that sends $\mu_{\mathcal{P}}$ to $H\backslash K(\mathcal{P})$ where H is some cocompact torsion free subgroup of $G_{\mathcal{P}}$). The later set is countable so theorem 3.0.1 follows immediately. □

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Vita

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